PROPERTIES OF MATTER

AND

HYDROSTATICS

PROPERTIES OF MATTER

AND

HYDROSTATICS

PROPERTIES OF MATTER

AND

HYDROSTATICS

(For Degree Classes)

D. S. SUBRAHMANYAM, M. A., B. Ed.
K. S. R. BHASKARA SARMA, M. SC.

197/ROD

TRIVENI PUBLISHERS
GOKHALE HALL, 9, ARMENIAN STREET
MADRAS-1

First Edition: July 1959
Fifth ,, June 1969

532.2
SUB

PRICE
Rupees Seven and Fifty Paise

Printed at their own Printing Works

PREFACE TO THE FIRST EDITION

The introduction of the Three-Year Degree Course and the consequent revision of syllabus in Physics for the Degree classes has produced a need for a book on Properties of Matter and Hydrostatics which will satisfy the new requirements of students. We have made an attempt to give a clear and concise treatment of the subjects dealt with. We have also included a chapter on Hydrostatics which was hitherto being done along with Mechanics. A chapter on Introductory Calculus is also appended at the end for the benefit of students who join the First year of the Degree class without any knowledge of the same. We trust that the book will be of substantial help and guidance to the students of the Degree classes.

We take this opportunity to thank Sri P. V. S. Mukteswaram, M. Sc., Lecturer in Physics of our college for his sustained help in bringing out the book. We also thank Sri N. Ch. Pattabhiramacharyulu, M. A., Lecturer in Mathematics of our college who has gone through the chapter on Calculus and made valuable suggestions.

Suggestions for improvement will be received with thanks.

—Authors

PREFACE TO THE FIFTH EDITION

We express our grateful thanks to our friends who offered their valuable suggestions.

Suggestions for the improvement of the book will be gratefully received and acknowledged with thanks.

—Authors

CONTENTS

Chapter		*Page*
I	Units and Measurements	1
II	Circular Motion	21
III	Simple Harmonic Motion	40
IV	Elasticity	64
V	Gravitation	114
VI	Kinetic Theory	148
VII	Diffusion and Osmosis	167
VIII	Compressibility	193
IX	Surface Tension	204
X	Viscosity	247
XI	Hydrostatics	270
	Introductory Calculus	(i) to (xlv)
	Index	

Chapter I

UNITS AND MEASUREMENTS

A. UNITS AND DIMENSIONS

1. For an exact description of any physical quantity, we shall have to specify (1) a standard or a unit of the same kind in terms of which the quantity can be estimated and (2) the number of times the quantity contains the unit. This numerical relation of the quantity to the unit is called the **measure** of the quantity in terms of the unit.

For instance, the length of a rod may be stated as 12 feet. Now *foot* is the unit of measurement and 12 is the measure of the quantity which is length. The same length may also be expressed as 4 yards or 144 inches. The measure of a quantity, therefore, depends upon the unit adopted, being smaller for a larger unit and larger for a smaller unit. In other words, *the measure varies inversely as the standard or the unit adopted.*

2. Fundamental and derived units

Throughout the whole range of Physics, we come across a number of physical quantities, each one having a unit of its own. These units may be chosen quite arbitrarily, having no relation between one another. But it will be convenient to adopt a set of units which are in some way related to a few fundamental units. The units of mass, length and time are, by convention, chosen as the fundamental units and all other units adopted for measurement in Physics, especially in Mechanics, are derived from these fundamental units.

Thus, we may define unit area as the area of a square having a side of unit length; unit velocity as that of a body which covers a distance equal to unit length in unit time; unit acceleration as that of a body which undergoes a change of unit velocity in unit time; and unit force as that which acting on a body of unit mass produces unit acceleration in its own direction.

Such units of area, velocity etc., are called **derived units.**

If, now, the *foot* is chosen as the unit of length, the unit of area becomes a *square foot.* But if the *yard* is taken as the unit of length, the unit of area becomes a *square yard,* which is nine times as large as a square foot. Thus, the value of a derived unit depends upon the value of the fundamental unit or units from which it is derived.

3. Systems of units

Two systems of units are in general use: The Metric (or C. G. S.) system and the British (or F. P. S.) system.

On the Metric (or C. G. S.) system of measurement, the units of length, mass and time are the *centimetre, gramme, and second* respectively.

On the British (or F. P. S.) system, the *foot, pound* and *second* are the standards of length, mass and time respectively.

The other units of measurement can be derived from these fundamental units.

For all scientific work, the C. G. S. system of units is preferred. It is well-known that, in our country, we are trying to introduce this system for all purposes of measurement.

4. Derived quantities and dimensions

Quantities like area, volume, density etc., are called **derived quantities,** for they can be derived from the three fundamental quantities: length, mass and time. For example, the area of a rectangle of length 10 cm., and breadth 5 cm., is 10 cm. $\times$ 5 cm. $=50$ cm.^{2} or 50 sq. cm. The volume of a rectangular block of length 10 cm., breadth 5 cm. and height 4 cm. is 10 cm. $\times$ 5 cm. $\times$ 4 cm. $=200$ cm.^{3} or 200 cubic centimetres. It is seen that the unit of length occurs to the second power in measurement of area and to the third power in measurement of volume. If the unit of length be denoted by L, the unit of area is L^2 and that of volume is L^3. The dimensions of area are then said to be $[L^2]$ or two in length and of volume, $[L^3]$ or three in length.

It may be noted that the dimensions of a physical quantity illustrate the nature of the quantity as apart from its magnitude. The nature of an area is different from that of a length. But, we also know that an area is obtained by multiplying one length with another length. Thus, if we say that a length is of one dimension in length and denote it by [L], an area is of two dimensions in length and it may be denoted by $[L^2]$. Similarly, any volume is of three dimensions in length and it is denoted by $[L^3]$. The difference in the nature of the physical quantities is indicated by the difference in their respective dimensions.

Now, velocity is obtained by dividing length by time; it is, in dimensional notation, [L/T] or $[LT^{-1}]$ and it is called the dimensional formula for velocity. Thus the dimensions of velocity are 1 in length and -1 in time. The dimensions of some of the physical quantities are given below:

Physical Quantities	*Dimentional formulae*	*Dimensions in* Length	Mass	Time
Length	[L]	1	–	–
Area	$[L^2]$	2	–	–
Volume	$[L^3]$	3	–	–
Velocity	[L/T] or $[LT^{-1}]$	1	–	–1
Acceleration (Velocity/Time)	$[L/T^2]$ or $[LT^{-2}]$	1	–	–2
Force (Mass × Acceleration)	$[M.\ L/T^2]$ or $[M.\ LT^{-2}]$	1	1	–2
Momentum or Impulse (Mass × Velocity)	[M. L/T] or $[MLT^{-1}]$	1	1	–1
Energy or work (Force × Displacement)	$[ML^2/T^2]$ or $[ML^2T^{-2}]$	2	1	–2
Power (Work / Time)	$[ML^2/T^3]$ or $[ML^2T^{-3}]$	2	1	–3

It may be seen, in this connection, that the dimensions of kinetic energy which is equal to $[\frac{1}{2} \times \text{mass} \times (\text{velocity})^2]$ can be deduced as follows:

Dimensions of a numerical constant, $\frac{1}{2}$		....	zero
Dimensions of mass	....	....	[M]
Dimensions of (velocity)^2	....	$\left[\frac{L}{T}\right]^2$	$= [L^2/T^2]$

Thus combining these expressions, we have for kinetic energy the dimensions $[ML^2/T^2]$ or $[ML^2T^{-2}]$. These are the same as for work, a result which is to be expected from the fact that they are mutually convertible.

5. Dimensional equations

When we have an equation expressing a relation between certain physical quantities, the dimensions of the physical quantities on the two sides of the equation must agree. When the physical quantities on each side of the equation are expressed in their dimensions, the resulting equation is called the Dimensional Equation.

(*a*) *Dimentional equations may be used for checking formulae*, by taking into account the dimensions of the quantities on either side of the equation.

For instance, we have the equation for the period of oscillation of a simple pendulum, $T = 2\pi\sqrt{\frac{l}{g}}$.

Considering the left hand side of the equation, the dimensions are [T].

On the right hand side, the dimensions of 2π are zero and the dimensions $(l/g)^{\frac{1}{2}}$ are $[L]^{\frac{1}{2}}/[L\cdot T^{-2}]^{\frac{1}{2}} = [T]$.

Hence the dimensional equation becomes $[T] = [T]$, a result which is in agreement with the relation deduced.

(*b*) ***Dimensional equations may also be used for converting a quantity from one system of units to another system of units.*** The measure of any physical quantity (i.e., the numerical value by which a unit has to be multiplied to express the quantity) is inversely proportional to the magnitude of the unit adopted for measurement. Thus a rod is 12 or 4 or 144 units according as the unit is the foot or the yard or the inch respectively.

Suppose that any physical quantity of dimensions, a in length, b in mass and c in time has a measure n when measured in a system having L, M and T as the units of measurement. The quantity will then have a value $n \times [L^a M^b T^c]$. Let the units of measurement be now changed to L_1, M_1 and T_1. The dimensions of the quantity are not altered and there will be a change only in the measure of the quantity. Let the measure in the new system be n_1.

Then $n \times [L^a M^b T^c] = n_1 \times [L_1^a M_1^b T_1^c]$ or the measure in the new system $n_1 = n \times [L^a M^b T^c] / [L_1^a M_1^b T_1^c]$.

Let us, as an example, convert an acceleration of 32 feet per second per second (the acceleration of a freely falling body) into centimetres per minute per minute.

Then $n_1 = 32 \times [LT^{-2}] / [L_1 T_1^{-2}] = 32\ [L/L_1]\ [T_1 / T]^2$.
But L = 1 foot = 12×2.54 cm. = 30.48 cm. = $30.48 L_1$.
$\therefore$ $L/L_1 = 30.48$.
$T_1 = 1$ minute = 60 seconds = 60 T
$\therefore$ $[T_1/T] = 60$.
$n_1 = 32\ [30.48]\ [60]^2 = 35.12 \times 10^5$.

(*c*) *Dimensional equations may, in some cases, be used for deriving formulae* as can be seen from the following problem :

Problem : *Show that when a body is dropped to the ground, the relation for the height S covered in a time t will be of the form* $S = kgt^2$ *where k is a numerical constant and g, the acceleration due to gravity.*

The distance covered depends upon the acceleration due to gravity to which the falling body is subject and the time.

We may, therefore, write the relation as

$$S = k.g^x.t^y.$$

The values of x and y may be obtained from the corresponding dimensional equations.

The dimensions of	S	....	[L]
,, ,, ,,	k	....	zero
,, ,, ,,	g	....	$[LT^{-2}]$
and ,, ,, ,,	t	....	[T]

We get the dimensional equation

$$[L] = [LT^{-2}]^x [T]^y$$
$$= [L]^x [T]^{-2x+y}.$$

Now equating the dimensions of [L] on the two sides,

$$1 = x \text{ or } x = 1;$$

and equating the dimensions of T on the two sides

$$0 = -2x + y \text{ or } y = 2x = 2 (\because x = 1)$$
$$\therefore S = k.g.t^2.$$

But the dimensional method of establishing relations between physical quantities is not without its limitations. It will not give us the numerical relation between the quantities involved. This is seen in the example cited above. We know that $S = \frac{1}{2} gt^2$ but we do not get the numeric $\frac{1}{2}$ in this method of treatment.

Further, this method fails when there are more than three physical quantities on any side of the equation since we can have only three equations formed corresponding to the dimensions of the three fundamental quantities—Length, Mass and Time.

6. Choice of standard units

The choice of a fundamental unit is extremely important and it should satisfy certain requirements. *The unit should easily be reproducible at any place and should be constant.*

Unit of length: On the C. G. S. system, the unit of length is the *centimetre*. It is equal to one-hundredth part of the length between two marks on a certain platinum-Iridium bar kept at 0°C in the Archives in Paris. The length between the marks, therefore, becomes a metre (Nowadays, the unit is defined in terms of the wavelength of a certain light in vacuum under normal conditions).

On the F. P. S. system, the standard of length is the *foot*. It is one-third of the standard yard, which is defined as the distance between two marks on a certain bronze bar, at a temperature of 62°F which is preserved by the British Board of Trade.

Unit of mass: On the C. G. S. system, the *gramme* is the unit of mass. Sometime back, it was defined as the mass of a cubic centimetre of water at 4°C. But it is now defined as

one-thousandth part of the mass of a certain lump of platinum-iridium, kept in the French Archives in Paris.

On the F. P. S. system, the unit is a *pound* which is the mass of a lump of platinum kept by the British Board of Trade.

Unit of time: The *second* is the unit of time in both the systems of units. It is defined as $\left(\frac{1}{24} \times \frac{1}{60} \times \frac{1}{60}\right) =$ (1/86,400) of the *mean solar day.*

The interval between the noon on one day and the noon on the next day is called the *solar day.* But the solar day is not of constant duration as it varies continuously from day to day in the year. The average of the solar days taken over a year is called the **mean solar day.** The *second,* the unit of time, is 1/86,400 of the mean solar day.

B. MEASUREMENTS

7. Measurement of Length

A scale divided into centimetres and millimetres (or one divided into inches and fractions of an inch) is generally used to measure lengths. The lengths measured with a millimetre scale will be accurate to a millimetre. Similarly, measurements made with a foot rule, divided into inches and tenths of an inch, will be correct to the tenth of an inch. Special instruments are therefore needed for making measurements of higher accuracy. Two such instruments, the vernier calipers (which is based on the principle of the vernier) and the screw gauge (which is based on the principle of the screw) are described in our "Physics for Pre-University Students" which may be referred to.

8. Spherometer

The Spherometer is another instrument for accurate measurement of length, which also works on the principle of the screw. It is used for measuring thicknesses of small objects and for determining the radius of curvature of a spherical surface.

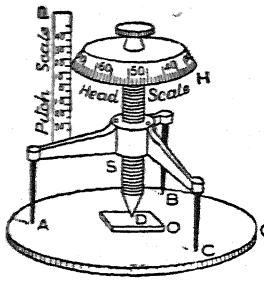


Fig. 1

The spherometer consists of a rigid metal frame resting on three legs ending in sharp points A, B, C. A screw, also ending in a sharp point D, works through a nut at the centre of the frame carrying the legs. The head of the screw carries a circular disc, H. The edge of the disc is divided into a known number of equal parts (usually 100). The pitch scale P is fixed to the frame so that it is parallel to the axis of the screw and stands by the side of the disc. The pointed ends of the legs lie at the vertices of an equilateral triangle and the pointed end of the screw coincides with the circum-centre of that triangle, when lowered to be in the plane of the pointed ends of the legs.

Least count of the Spherometer: The screw of the spherometer is turned through a known number of revolutions (say 10) and the distance (s) by which the circular disc advances along the pitch scale is noted. The pitch of the screw, p (i.e., the distance through which it advances for a complete turn $= s/10$) is then calculated. The pitch of the screw of spherometers commonly used in the laboratories is 0·5 mm.

The least count (L) of the instrument is then calculated by dividing the pitch of the screw (p) by the number of divisions (n) on the circular disc. The number of divisions on the circular disc is usually 100 so that the least count of a spherometer in common use is 0·005 mm.

9. To determine the thickness of an object: The spherometer is placed on a plane glass plate, G. The screw is first raised sufficiently and the object, O, is introduced under the screw. The screw is next lowered until it just touches the object as indicated by *the rotation test.*

A gentle tap is given laterally to one of the legs of the spherometer from time to time as the screw is lowered. At a certain stage, the spherometer begins to rotate about the screw. This happens after the point of the screw touches the plate. The *position in which the spherometer just begins to rotate about the screw as it is lowered is taken as (Rotation Test) the required position.*

The division (s_1) on the circular disc which coincides with the edge of the pitch scale is noted. The spherometer is then bodily raised, the object removed from the glass plate and the spherometer once again placed back on the glass plate. The screw is next lowered down gently until the screw just touches the plate as indicated by the same rotation test as described above, counting the number of complete revolutions, (m) given to it. The division (s_2) on the circular disc against the pitch scale noted.

The thickness of the object is obviously equal to the distance through which the screw has been lowered. It is therefore equal to

$$[m.p + (s_2 - s_1)L]$$

where p is the pitch of the screw and L, the least count.

10. To determine the radius of curvature of a convex surface, say of a watch glass: The watch glass is placed on a plane sheet of glass, with the convex surface upwards. The spherometer is placed over the surface so that its three legs rest on the surface and it is adjusted so that the pointed end of the screw just touches the spherical surface. The reading s_1 on the circular scale is taken. The instrument is bodily raised, the watch glass removed and the spherometer once again placed on the glass plate. The screw is now lowered until its point just touches the plane glass plate. The number of revolutions, m, made by the screw and the final reading s_2 on the disc are noted as before. Then the height, h, of the spherical surface is equal to $[m.p+(s_2-s_1)L]$.

The spherometer is next pressed on a plane sheet of paper and the distance between the legs (i.e., impressions) are measured. If l be the mean of these three values (which will be nearly equal), the radius of curvature of the spherical surface R is given by the relation,

$$R = \frac{l^2}{6h} + \frac{h}{2}$$ as shown below.

Let A, B, C (Fig. 2) represent the three pointed ends of the legs which form the vertices of an equilateral triangle. Let D be the circum-centre of the triangle ABC, where the pointed end of the screw meets the plane of the triangle when lowered down.

Now that the triangle is equilateral,

$AB = BC = CA = l$.

In the $\triangle$ ABE, $\angle B = 60°$

$$\therefore AE = AB \text{ Sin } 60° = l. \frac{\sqrt{3}}{2}.$$

But $AD = 2/3\ AE.$

Hence $AD = \frac{2}{3}. \frac{\sqrt{3}}{2}. l = \frac{l}{\sqrt{3}}.$

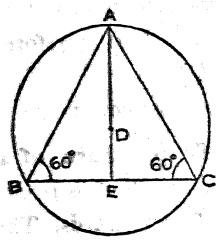


Fig. 2

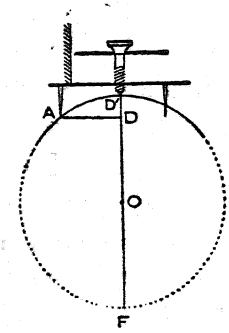


Fig. 3

Let Fig. 3 represent central section of the sphere (of which the surface is a part) containing the point, A, of one of the legs and the point of the screw, D'. The axis of the screw passes through D (the circum-centre of the triangle formed by the feet of the legs) and O (the centre of the sphere) and cuts the sphere again at F. Now $D'D = h$ and $DF = (2R - h)$.

Then from the properties of circles we have,

$$AD^2 = D'D \times DF \text{ or } \left(\frac{l}{\sqrt{3}} \right)^2 = h\ (2R - h).$$

$$\text{Hence } R = \frac{l^2}{6h} + \frac{h}{2}.$$

R is calculated making use of this relation.

11. Radius of curvature of a concave spherical surface : In the case of a concave surface, readings of the spherometer are obtained placing it first on the plane glass plate and then on the concave spherical surface. As before, if s_1 be the first reading on the circular scale, m, the number of revolutions given to the screw and s_2 the final reading on the concave surface, the depth of the concave surface, $h = [m.p + (s_2 - s_1)\ L]$. Obtaining h in this way and measuring l as before, the radius of curvature R of the spherical surface can

be calculated from the relation, $R = \frac{l^2}{6h} + \frac{h}{2}$.

This relation can be derived in the same manner as in the case of a convex spherical surface.

Measurement of Mass

12. The mass of a body is the quantity of matter contained in it and it can be determined by comparing its weight with that of a set of known weights in a common balance.

Common Balance

A common balance used in laboratories consists of a hollow rigid beam AB, provided with a fine knife-edge F at its centre. The beam

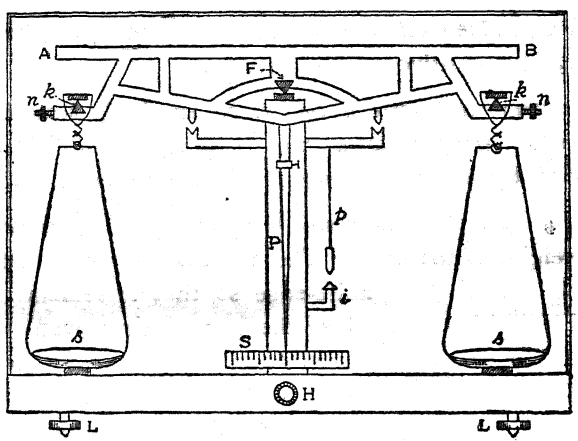


Fig. 4

turns freely about the knife-edge F, which rests on two agate plates fixed at the top of a sliding rod which works through a hollow pillar fixed centrally on a wooden base board. A handle H in front of the base board helps in raising or lowering the sliding rod and along with it, the beam supported on it. Scale pans, *s*, *s*, are suspended by means of stirrups, *k*, *k*, from knife-edges fixed at the two ends of the beam. When the beam is in the lowered position, it will not swing and it is said to be arrested. When the beam is arrested, the central knife-edge is off the agate plates and the stirrups, off the knife-edges at the ends. The knife-edges will not be thus supporting any weight when the beam is arrested. By arresting the beam when

the balance is not in use, the wear and tear of the knife-edges is reduced and the life of the balance prolonged. A long pointer P fixed at the middle of the beam, moves over a short ivory scale, S, fixed at the bottom of the pillar. The pointer is regulated to move within the range of the scale by working the nuts *n*, *n* provided at the ends of the beam.

The base board is levelled by working the levelling screws, *L*, *L*, on which it rests. They are worked so that the plummet *p* stands vertically above the index, *i*.

The balance is usually enclosed in a glass case which is provided with a sliding door on the front side in order to protect it from draughts of air.

13. Requisites of a good balance: A good balance should be (a) true, (b) sensitive and (c) stable.

(a) **Truth:** *A balance is said to be true if the beam is horizontal when the pans are empty or when they are equally loaded.* The conditions to be satisfied by a true balance may be deduced as follows:

Let a and b be the lengths of the arms of a balance and S_1 and S_2 be the weights of the pans. The beam is supported on the central knife-edge so that it remains horizontal when the pans are removed. *The centre of gravity of the beam* then *lies vertically below the knife-edge F* and the weight of the beam exerts no moment about the fulcrum, F, when the beam is horizontal. If the pans are suspended from the ends of the beam and the beam remains horizontal again, we have, on taking moments about F,

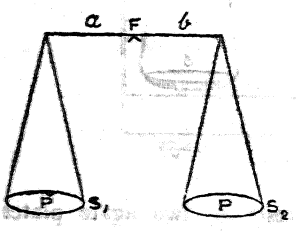


Fig. 5

$$S_1 \cdot a = S_2 \cdot b \ldots\ldots (1)$$

Let two equal weights P and P be now placed in the two pans and let the beam again remain horizontal.

Taking moments about F again, we have

$$(S_1 + P)a = (S_2 + P)b \ldots\ldots (2)$$

Subtracting (1) from (2) we get,

$P \cdot a = P \cdot b$ or $a = b$ and from (1), $S_1 = S_2$.

In the case of a true balance threfore, (1) *the arms are of equal length* and (2) *the pans are of equal weight.*

(b) **Sensitiveness :** *A balance is said to be sensitive when the beam deflects appreciably from the horizontal position for any difference between the weights in the pans, however small the difference may be.*

A common balance should be true in the first instance and it should satisfy the two conditions stated above. We may now investigate the conditions to be satisfied by it if it is to be also sensitive.

Consider a vertical section taken through the centre of the beam AB of a balance and let the section cut the knife-edges in A′, F and B′.

Let l be the length of the arms;

d, the depth of the centre of gravity of the beam below the fulcrum F.

h, the height of the knife-edges at A and B above F when the beam is at rest and horizontal,

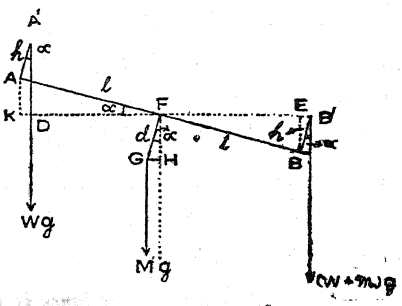


Fig. 6

M, the mass of the beam,

W, the mass of the left pan and the load in it; and

(W + m) the mass of the right pan and the load in it.

The excess weight 'm' on the right side deflects the beam from its horizontal position. Let the beam be inclined at an angle α to the horizontal.

When the beam remains in equilibrium under the action of the different forces acting on it, the algebraic sum of the moments of the forces about F should be zero.

$$Wg\,(DF) + Mg.\ GH = (W + m)g.B'F.$$

Now $DF = KF - KD$

$$= l\cos\alpha - h\sin\alpha$$

$$GH = d\sin\alpha$$

$$B'F = B'E + EF$$

$$= h\sin\alpha + l\cos\alpha.$$

$$Wg\,(l\cos\alpha - h\sin\alpha) + Mg.d\sin\alpha$$
$$= (W + m)\,g\,(l\cos\alpha + h\sin\alpha)$$

When α is small, $\cos\alpha = 1$ and $\sin\alpha = \alpha$ so that

$$W(l - h\alpha) + Md.\alpha = (W + m)\,(l + h\alpha)$$

or $Wl - Wh\,\alpha + Md\,\alpha = Wl + Wh\,\alpha + ml + mh\,\alpha.$

Neglecting $mh\,\alpha$ and rearranging the terms, we have

$$\alpha / m = l / (Md - 2Wh).$$

The ratio of the deflection, α, to the difference, between the loads in the two pans, m, may be said to define the sensitiveness of the balance. It is obvious that it will be independent of the load in the pans if $h = 0$ (i.e., if the three knife-edges are in the same plane) and increases with load if h is positive and decreases with load if h is negative.

It is now clear from the expression for the sensitiveness that increase in the sensitiveness can be secured by (*1*) *increasing the length of the arms* (*2*) *decreasing the weight of the beam and* (*3*) *decreasing the depth of the centre of gravity of the beam below the fulcrum.*

Common balances employed in laboratories are highly sensitive and it may be noted that *the beams are purposely made hollow in order to reduce the weight* and thereby increase their sensitiveness.

(c) **Stability**: *A balance is said to be stable if the beam return to its equilibrium position quickly when slightly disturbed.* When a balance in equilibrium is disturbed, it is the moment of the weight of the beam about the fulcrum, $Mgd\sin\alpha$, that tends to restore it to equilibrium. Stability will therefore be secured in a balance if (*1*) *M, the mass of the beam and* (*2*) *d, the distance of the centre of gravity of the beam below the fulcrum are made large.* These two conditions for stability are opposed to the conditions for securing sensitiveness. The purpose to which a balance is to be put will decide whether a balance should have more sensitiveness or greater stability and the values of M and d are chosen accordingly.

14. Weighing by the method of oscillations

The beam of a sensitive balance, when in action, will swing for a long time, when nearly equal weights are kept in the pans and it will be inconvenient to wait till the beam comes to rest in order to know whether the weights are equal. The resting point in all sensitive

balances is therefore determined by noting down an odd number of consecutive turning points of the pointer on the scale. This method is called **weighing by oscillations.**

Keeping no load in the pans, the balance is released by turning the handle, H. The pointer then begins to swing. The first few swings are ignored and five consecutive turning points of the pointer on the scale are noted. For purposes of calculating the resting point, it will be more convenient if the graduations on the scale are taken to have their *zero on the extreme left of the scale.* Usually three turning points are obtained on the left and two on the right. The average of the turning points on the left and the averge of those on the right are calculated. The mean of the two averages gives the resting point of the pointer on the scale. The resting point obtained with zero load in the pans is called the **zero resting point of the balance** (*Z. R. P.*)

For instance, if in any experiment, the consecutive turning points are 5.2, 13.5, 5.8, 13.0, 6.0, the observations are arranged and the Z. R. P. is calculated as follows :

	5.2	13.5
	5.8	13.0
	6.0	
	17.0	26.5
Mean...	5.66	13.25

$$\text{Z. R. P.} = \frac{5.66 + 13.25}{2} = 9.455 = 9.5.$$

To find the weight of a body : The given body is placed in the left pan, and the weights in the right pan. The weights in the right pan are adjusted so that when the beam is raised, the pointer swings within the scale and almost equally on either side of the Z. R. P. The resting point is again determined. If this resting point is lower than the Z. R. P., the weights in the right pan are heavier and if it is higher than the Z. R. P., the weights are lighter. The weight in the right pan is then decreased or increased dy 10 *mg.* as the case may be and the resting point determined again. This resting point should be on the other side of the Z. R. P. Thus we get two resting points on

either side of the Z. R. P. corresponding to two weights placed in the right pan, which differ from each other by 10 *mg*.

The Z. R. P. corresponds to the horizontal position of the beam. If the weights in the right pan are to be equal to the weight of the body, the resting point obtained should be equal to the Z. R. P.

Of the two weights, that, which gives a resting point nearer to the Z. R. P., is the more correct one of the two and it is taken as the weight of the body correct to a *centigram*. The weight of the body may also be calculated correct to a milligram from the observations obtained, as shown in the following example :

Resting point with no load in the pans		= 9.5
Resting point with body in the left pan and 15.66 gm. in the right pan	}	= 13.2
Resting point with body in the left pan and 15.67 gm. in the right pan	}	= 8.9
Difference in R. Ps. when the weight is increased by 0.01 gm. = 13.2 − 8.9		= 4.3
Difference between the higher R. P. and the Z. R. P. = 13.2 − 9.5		= 3.7

Corresponding difference of weight required
$= 3.7 \times 0.01/4.3 = 0.0086$ gm.

The higher R. P. is given by a weight that is lighter than the correct weight.

∴ The correct weight = 15.66 + 0.0086 = 15.6686 or 15.669 gm.

15. Ratio of arms and the true weight of a body with a false balance.

If the arms of a balance are not equal in length, the balance is said to be **false**. Gauss devised a method by which the correct weight of a body may be obtained even with a false balance. (It is assumed that the beam remains horizontal when there is no load in the pans and that the centre of gravity of the beam lies below the fulcrum.) The body is weighed twice keeping it first in the left pan and then in the right pan.

Let W_1 and W_2 be the weights obtained in the two cases

respectively. If l_1 and l_2 are the lengths of the arms and if W is the true weight of the body, we have,

when the body is placed in the left pan,

$$W. l_1 = W_1. l_2 \ \ldots \ \ldots \ (1)$$

and when the body and the weights are interchanged

$$W_2. l_1 = W. l_2 \ \ldots \ \ldots \ (2)$$

From (1) and (2)

$$W = \sqrt{W_1. W_2}$$

and $l_1/l_2 = \sqrt{W_1/W_2}$.

The true weight of the body as well as the ratio of the arms of the balance can be thus obtained by the method of double weighing.

Examples

1. *Given that the velocity, v, of a capillary wave depends upon*
   (a) ρ, *the density of the liquid,*
   (b) T, *the surface tension and*
   (c) λ, *the wave length of the wave,*

establish the relation $v \propto \sqrt{T/\rho\lambda}$.

Let $v = k. \rho^{\alpha}. T^{\beta}. \lambda^{\gamma}$, k being numerical constant.

Dimensions of $v = LT^{-1}$

„ $\rho = ML^{-3}$

„ $T = \dfrac{\text{Force}}{\text{Length}} = \dfrac{MLT^{-2}}{L} = MT^{-2}$,

„ $\lambda = L$

„ $k = 0$

Hence we have,

$$[LT^{-1}] = [ML^{-3}]^{\alpha}. [MT^{-2}]^{\beta}. [L]^{\gamma}.$$

Now equating the dimensions of [L] on the two sides.

$$1 = -3\alpha + \gamma \ \ldots \ \ldots \ \ldots \ \ldots \quad (1)$$

Equating the dimensions of [M]

$$0 = \alpha + \beta \text{ or } \alpha = -\beta \ \ldots \ \ldots \ \ldots \quad (2)$$

Finally equating the dimensions of [T] on the two sides

$$-1 = -2\beta \text{ or } \beta = \tfrac{1}{2} \ \ldots \ \ldots \ \ldots \quad (3)$$

From (2) $\alpha = -\beta$ or $\alpha = \frac{1}{2}$ ... ... (4)

From (1) $-3\alpha + \gamma = 1$ or $\frac{3}{2} + \gamma = 1$

i.e., $\gamma = -\frac{1}{2}$ ... (5)

$\therefore v = k.\rho^{-\frac{1}{2}}.T^{\frac{1}{2}}.\lambda^{-\frac{1}{2}}$ or $v \propto \sqrt{T/\rho\lambda}$.

2. *The kinetic energy of a body is 500 ft. poundals in the F. P. S. system. Find its value in the C. G. S. system given that 1 foot = 30·48 cm. and 1 lb. = 453·6 gm.*

Let x be the measure in the C. G. S. system.

Then $x\,[E_1] = 500\,[E]$.

Or $x\,[M_1L_1{}^2T_1{}^{-2}] = 500\,[ML^2T^{-2}]$

$$\therefore x = 500\,\frac{[ML^2T^{-2}]}{[M_1L_1{}^2T_1{}^{-2}]}$$

Now $[M] = 453{\cdot}6\,[M_1]$

$[L] = 30{\cdot}48\,[L_1]$

$[T] = [T_1]$

Hence $x = 500 \times 453{\cdot}6 \times (30{\cdot}48)^2$

$= 2{\cdot}107 \times 10^8$ ergs.

Exercises

1. Find the dimensions of force, energy, power, and pressure.

2. Given that the velocity of sound in a gas (v) is a function of its pressure (p) and density (d), prove that $v \propto \sqrt{p/d}$.

3. Show that the acceleration 'a' of a particle moving with a uniform speed v round a circle of radius r is proportional to v^2/r.

4. Convert by the method of dimensions (1) 4·2 Joules to foot-pounds. (2) Horse-power into watts. (3) Dynes per sq. cm. into poundals per sq. ft.

Ans.: (1) 3·097 (2) 746 and (3) $6{\cdot}719 \times 10^{-2}$.

5. Assuming the unit of length to be one mile and the unit of time to be one minute, find the values of the units of velocity and acceleration, in terms of the fundamental units in F. P. S. system.

Ans.: 88 ft. per sec.; $\frac{22}{15}$ foot-second units.

6. Describe an experiment to determine the radius of curvature of a spherical surface with a spherometer.

7. Describe a common balance and explain how it may be used to determine the mass of a body correct to a milligram.

8. What are the requisites of a good balance? Explain how they are secured in practice.

9. Describe how a false balance may be used to determine the correct weight of a body.

10. Show that the sensitiveness of a balance is independent of the load in the scale-pans when the terminal knife-edges are in the same plane as the central knife-edge.

11. Given that the frequency of vibration, n, of a stretched string is a function of the tension T, the length l and the mass per unit length of the wire m, show that

$$n \propto \frac{1}{l}\sqrt{\frac{T}{m}}.$$

12. Deduce the dimensions of η from the relation

$$\eta = \frac{\pi P a^4 . t}{8 v l}$$

η being the coefficient of viscosity of a liquid; P, the pressure difference between the ends of a capillary tube through which the liquid flows; a, the radius of cross section of tube; l, the length of the tube; v, the volume of the liquid flowing out of the tube and t, the time of flow in seconds.

13. Assuming that the force of gravitational attraction F between two masses M_1 and M_2 separated by a distance d is given by the relation,

$$F = G.\frac{M_1 M_2}{d^2},$$

find the dimensions of the gravitational constant G.

14. Verify the correctness of the following relations:

(a) $T = 2\pi\sqrt{\frac{k^2 + l^2}{lg}}$ (b) $F = 6\pi a \eta v$

(c) $v^2 = u^2 + 2as.$

15. What do you mean by the dimension of a physical quantity? What is a dimensional equation? What are the limitations of such

an equation? Mention its uses. Deduce the dimensions of (1) constant of gravitation, (2) Young's modulus, (3) surface tension and (4) coefficient of viscosity. (*A. U. April 1962*)

16. Define the dimensions of a physical quantity. Find the dimension of (a) force, (b) surface tension and (c) momentum in terms of frequency, velocity and density as fundamental units.

Test by means of the dimensional method the correctness of the equation $n = \frac{1}{2l}\sqrt{\frac{T}{m}}$ where 'n' is the frequency of the stretched wire, 'l' its length, 'T' its tension and 'm' its mass per unit length. (*A. U. April, 1963*)

17. Using the dimensional method find an expression for the velocity of sound in air, if it depends upon the atmospheric pressure and the density of air.

Using the same method show that, if a body moves in a straight line with an acceleration R/s^2 towards a fixed point in a straight line, s being the distance of the moving body from the fixed point, the time taken by the body to move to the centre of the force, starting from rest at a distance 'd' from it is proportional to $d^{\frac{3}{2}}R^{-\frac{1}{2}}$ (*A. U. Sept., 1963*)

18. What are fundamental and derived units? Give two examples of derived units deducing their relation with the fundamental units.

Show how the optical lever may be used to find the thickness of a small glass plate. (*A. U. March 1964*)

19. Describe the spherometer. Show how it may be used (a) to determine the thickness of a small glass plate and (b) the radius of curvature of a convex surface. (*A. U. Sept., 1964*)

20. What do you understand by the dimensions of a physical quantity? Obtain the dimensions of viscosity in terms of M, L and T. (*A. U. April 1965*)

21. Explain the principle and working of the optical lever. (*A. U. April 1965*)

Chapter II

CIRCULAR MOTION

A. ROTATION

16. When a body is moving so that the points lying on a straight line in it are not displaced, it is said to **rotate.** The straight line passing through the points which are not displaced during the motion of the body is called the **axis of rotation.** The rotation of bodies, which retain constant shape, is studied in this chapter. Such bodies are called **rigid bodies.** The distance between any two points chosen at random in a rigid body remains constant.

Let OA and OB be any two lines drawn in a body which is rotating about a fixed axis perpendicular to the plane of the paper, passing through 'O'. When the body rotates, let OA be displaced to OA′ and OB to OB′ in a short interval of time δt. Then in the case of a rigid body, it is easily seen that $\angle AOA' = \angle BOB' = \delta\theta$ (say). Thus every line passing through O is displaced through the same angle, $\delta\theta$. This angle $\delta\theta$ is called the **angular displacement** of the body.

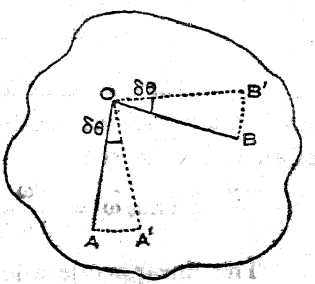


Fig. 7

If the angle $\delta\theta$ increases by equal amounts in equal intervals of time, the intervals (δt) being as small as we can make them, the body is said to *rotate with* **uniform angular velocity** which is usually denoted by the letter ω.

$$\therefore\ \omega = \delta\theta/\delta t \text{ and } \delta\theta = \omega \cdot \delta t \quad \ldots\ \ldots \qquad (1)$$

Hence the angular displacement =

(uniform angular velocity) × (time).

Let OA = r.

Then as the body rotates, A moves along a circle with centre O and radius 'r' so that $AA' = r.\delta\theta$.

If the velocity with which the point A is moving be called 'v',

$$v = \frac{AA'}{\delta t} = \frac{r.\,\delta\theta}{\delta t} = r.\,\omega. \qquad \ldots \quad \ldots \qquad (2)$$

It may be noted that since the value of 'r' differs from point to point, the linear velocities of the particles of a body differ from particle to particle and, for any particle the velocity is proportional to the distance of the particle 'r' from the axis. Since the motion of the particle is along a circle, the direction of motion and consequently the velocity of the particle are continuously changing. Its speed 'v' is, however, constant if its angular velocity, ω, is uniform.

If the angular velocity ω is not uniform, it may be defined in terms of the notation of Calculus to be $d\theta/dt$. i.e., the value that the ratio, between the increment in the angular displacement $\delta\theta$ and the interval of time δt, assumes as δt is made to approach the value zero, at the instant,

i.e., the angular velocity at any instant, $\omega = \dfrac{d\theta}{dt}$.

17. Angular acceleration: Following the analogy of linear motion, angular acceleration, $\dot{\omega}$, may be defined as the rate of change of angular velocity.

$$\text{Thus } \dot{\omega} = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}.$$

The linear acceleration $(\dot{v} = a)$ of the particle in the direction of motion is

$$\frac{dv}{dt} = \frac{d}{dt}(\omega r) = r\frac{d\omega}{dt} = r\dot{\omega} = r.\frac{d^2\theta}{dt^2}$$

$$\text{Thus } v = r.\frac{d^2\theta}{dt^2}.$$

18. Work done in rotation

Let a force F_1 act at A as the body is turned through a small angle $\delta\theta$ and A be displaced to A'. Since AA' is small, F_1 may be assumed to be constant during the displacement, so that the work done is equal to $F_1 \times AA' = F_1 \times OA \times \delta\theta$.

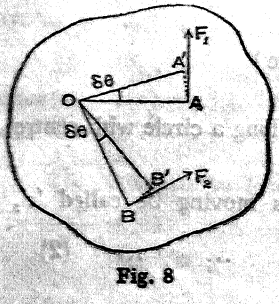


Fig. 8

But $(F_1 \times OA)$ is the moment of the force F_1 about O. Now the force F_1 at A gives rise to an equal and opposite force (the reaction) at O. But since O is not displaced, the work done by F_1 at

O is zero. The total work done by the two forces, therefore, is equal to $F_1 \times AA' = F_1 \times OA \times \delta\theta$.

Now, these two equal and opposite forces, each equal to F_1, constitute a couple the moment, C_1, of which is equal to $F_1 \times OA$, i.e., $C_1 = F_1 \times OA$.

$\therefore$ The work done by the couple due to F_1 as the body rotates $= F_1 \times OA \times \delta\theta = C_1 \times \delta\theta$.

Similarly, if F_2 be the force acting along BB′ at B, as B is displaced to B′ (in the rotation of the body through the angle $\delta\theta$), the work done by the couple due to F_2 at B and its reaction at $O = F_2 \times OB \times \delta\theta = C_2 \times \delta\theta$, where $C_2 = F_2 \times OB$.

Thus if F_1, F_2.....be the forces acting at different points of the body, and C_1, C_2 be the moments of the couples due to these forces, the work done on the body during the angular displacement, $\delta\theta$,

$W = (C_1 + C_2 + ...) \times \delta\theta$

$= C.\delta\theta$ (where $C = C_1 + C_2 + ...$ i.e., the algebraic sum of the moments of all the forces acting on the body which is equal to the resultant couple acting on the body).

Thus W, the work done in rotation, through a small angle $\delta\theta$. $\Big\}$ $= C \times \delta\theta$.

If the couple rotating the body be of constant value, the work done in rotating the body through an angle, θ is equal to $C\theta$.

If the body be rotated by a constant couple through one complete revolution,

the work done $= 2\pi C$ and if it makes n revolutions,
the work done $= 2\pi nC$.

19. Resultant couple and angular acceleration

If m_1 be the mass of the element at A (Fig. 8) on which the force acts $F_1 = m_1 \times a_1$, where a_1 is the linear acceleration of the particle.

$$\underline{\text{If } OA = r_1,\ a_1 = r_1 \cdot \frac{d^2\theta}{dt^2}}.$$

$$\therefore F_1 = m_1 r_1 \frac{d^2\theta}{dt^2}.$$

The moment of the force F_1 about $O = F_1 \times r_1 = m_1 r_1^2 \frac{d^2\theta}{dt^2}$.

Taking into consideration the equal force of reaction F_1 acting at O, the moment of which about O is equal to zero, the moment of the couple, C_1 due to F_1 will therefore be equal to $\left(m_1 r_1^2 . \frac{d^2\theta}{dt^2}\right)$.

Similarly, if $OB = r_2$ and if the mass of the element at $B = m_2$, the moment of the couple due to $F_2 = m_2 r_2^2 \frac{d^2\theta}{dt^2}$. Extending the treatment to the whole body, the algebraic sum of the moments of the couples acting on the body is equal to the resultant couple 'C' acting on the body and it is equal to $\left(m_1 r_1^2 \frac{d^2\theta}{dt^2} + m_2 r_2^2 \frac{d^2\theta}{dt^2} + \ldots\ldots\right)$

$$= \frac{d^2\theta}{dt^2} (m_1 r_1^2 + m_2 r_2^2 + \ldots\ldots)$$

$$= \frac{d^2\theta}{dt^2} . I,$$

where $I = (m_1 r_1^2 + m_2 r_2^2 + \ldots) = \sum m_1 r_1^2$.

The quantity I depends upon the masses of the particles and their respective distances from the axis of rotation. It is a constant for the body for a definite position of the axis of rotation and is called the **Moment of Inertia** of the body about that axis.

Thus $C = I \frac{d^2\theta}{dt^2} = I . \frac{d\omega}{dt}$.

(The resultant couple acting on the body = moment of inertia × angular acceleration.)

20. Kinetic energy of a rotating body

If the linear velocity of the element of mass m_1 at $A = v_1$, its kinetic energy $= \frac{1}{2} m_1 v^2{}_1 = \frac{1}{2} m_1 (r_1 \omega)^2 = \frac{1}{2} m_1 r_1^2 \omega^2$.

Since all the particles have the same angular velocity, ω, the kinetic energy of the whole body

$$= \tfrac{1}{2} m_1 r_1^2 \omega^2 + \tfrac{1}{2} m_2 r_2^2 \omega^2 + \ldots\ldots\ldots\ldots$$
$$= \tfrac{1}{2} \omega^2 (m_1 r_1^2 + m_2 r_2^2 + \ldots\ldots\ldots)$$
$$= \tfrac{1}{2} \omega^2 \sum m_1 r_1^2$$
$$= \tfrac{1}{2} I \omega^2.$$

21. Angular momentum or moment of momentum

The momentum of the element m_1 at A about O

$$= m_1 v_1 = m_1 r_1 \omega.$$

Moment of momentum of the element m_1 about O

$$= m_1 v_1 r_1 = m_1 r_1^2 \omega.$$

Similarly, the moment of momentum of m_2 about O

$$= m_2 r_2^2 \omega.$$

Thus the total moment of momentum of all the elements of the body $= m_1 r_1^2 \omega + m_2 r_2^2 \omega + \ldots\ldots\ldots$

$$= \omega \ (m_1 r_1^2 + m_2 r_2^2 + \ldots\ldots\ldots\ldots)$$

$$= \omega \ (\Sigma \ m_1 r_1^2) = I \omega.$$

22. A similarity can be seen between the corresponding expressions in translatory and angular motion from the following table. It is found that **moment of inertia plays the same role in angular motion as mass plays in translatory motion.**

Translatory motion	Angular motion
1. Displacement $= s$	Angular displacement $= \theta$
2. Velocity 'v' $= \frac{ds}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
3. Acceleration $a = \frac{d^2 s}{dt^2}$	Angular acceleration $\dot{\omega} = \frac{d^2\theta}{dt^2}$
4. Mass $= m$	Moment of Inertia $= I$
5. Momentum $= m \times v = m. \frac{ds}{dt}$	Angular momentum $= I\omega = I. \frac{d\theta}{dt}$
6. Force $F = m. \frac{d^2 s}{dt^2}$	Couple $C = I. \frac{d^2\theta}{dt^2}$
7. Work done $= F \times s$	Work done $= C \times \theta$
8. Kinetic energy $= \frac{1}{2} m v^2$ $= \frac{1}{2} m \left(\frac{ds}{dt} \right)^2$	Kinetic energy $= \frac{1}{2} I \omega^2$ $= \frac{1}{2} I \left(\frac{d\theta}{dt} \right)^2$

23. Radius of Gyration

The value of the moment of inertia of the body $I = \Sigma \ mr^2$ depends upon the mass of the body and the distribution of the mass about the axis of rotation. The moment of inertia of a body having a

mass M is also, therefore, written as $I = \sum mr^2 = MK^2$ where K is called the radius of gyration of the body about the axis. According to this expression, the moment of inertia of the body is equal to that of a particle which has the same mass as the body and which is situated at a distance K from the axis.

24. Moments of inertia in some simple cases

(i) Moment of inertia of a thin uniform rod about an axis passing through its middle point, perpendicular to its length:

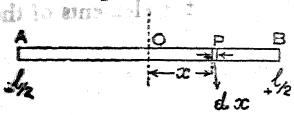


Fig. 9

Let the rod be of length l and of mass ρ per unit length.

Mass of the rod, $M = \rho l$ .... (1)

Consider a small element of the rod of length dx at a distance x from 'O', the mid point of the rod.

Mass of the element $= \rho\, dx$.

Moment of inertia of the element $= (\rho\, dx)x^2$.

Moment of inertia of the whole rod I is then obtained by integrating $(\rho.\, dx.\, x^2)$ between the limits $x = +l/2$ and $x = -l/2$.

$$\therefore I = \rho \int_{-l/2}^{+l/2} x^2 dx = \rho \left[\frac{x^3}{3}\right]_{-l/2}^{+l/2}$$

$$= \frac{l^3\rho}{12} = \frac{Ml^2}{12}$$

$$\therefore K^2 = \frac{l^2}{12}.$$

(ii) Moment of inertia of a circular disc about an axis passing through its centre perpendicular to the plane of the disc:

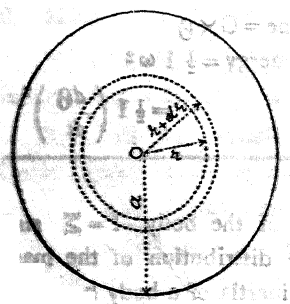


Fig. 10

Consider a thin annular strip of width dr at a distance 'r' from its centre O.

Its area $= \pi (r+dr)^2 - \pi r^2$

$= 2\pi r dr$

(neglecting higher powers of dr).

Let ρ be the mass per unit area of the strip.

Mass of the strip $= 2\pi r dr.\ \rho$.

Moment of inertia of the strip about 'O' $= (2\pi r dr.\rho)r^2 = (2\pi\rho r^3 dr)$

I, Moment of inertia of the disc $= \int_0^a 2\pi\rho r^3 dr = \frac{\pi a^4 \rho}{2}$ ....(1)

(where 'a' is the radious of the disc).

Now, if M is the mass of the disc, $M = \pi a^2 \rho$

$\therefore$ Substituting in (1) we get,

the moment of inertia of disc $= Ma^2/2$.

$\therefore K^2 = a^2/2$.

(iii) I, of a circular loop about an axis passing through its centre, perpendicular to the plane of the loop :

Since every element of the loop is at the same distance 'r' from the axis.

$I = \sum mr^2 = r^2 \sum m = Mr^2$ where M is the mass of the loop.

$\therefore K^2 = r^2$.

25. Theorem of Parallel Axes

If I is the moment of inertia of a body about any axis and I_g is the moment of inertia of the body about a parallel axis passing through its centre of gravity, $I = I_g + Ma^2$ where 'a' is the perpendicular distance between the two axes and M is the mass of the body.

Let I be the moment of inertia of the body about an axis passing through a point 'O' perpendicular to the plane of the paper and let a small element of mass dm be situated at P. The moment of inertia of the element, dm, about $O = dm.\ OP^2$.

Let G be a point at which a parallel axis passing through the centre of gravity of the body cuts the plane of the paper. Draw PQ perpendicular to OG (or OG produced).

$$I = \int dm.\ OP^2$$

$$= \int dm\ (OQ^2 + PQ^2)$$

$$= \int dm.\ OQ^2 + \int PQ^2.\ dm$$

$$= \int (OG + GQ)^2\ dm + \int PQ^2.\ dm$$

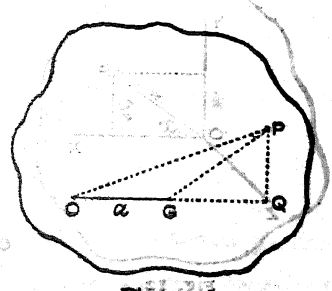


Fig. 11

$$= \int OG^2 \cdot dm + \int GQ^2 \cdot dm + 2 \int OG \cdot GQ \cdot dm + \int PQ^2 \cdot dm$$

$$= \int OG^2 \cdot dm + 2 \int OG.GQ \cdot dm + \int PG^2 \cdot dm$$

$$(\because GQ^2 + PQ^2 = PG^2)$$

Now, $2 \int dm.OG.GQ = 2.OG. \int GQ \cdot dm = 0$ since $\int GQ \cdot dm = 0$ as it is the resultant moment of the masses of different elements, of which the body is composed, about the centre of gravity.

$$\therefore I = \int OG^2 \cdot dm + \int PG^2 \cdot dm.$$

If $OG = a$, $I = Ma^2 + I_g$

where $I_g \left(= \int PG^2 \cdot dm \right)$, is the moment of inertia of the body about the parallel axis passing through the centre of gravity.

26. Theorem of perpendicular Axes

If OX, OY are two axes which are mutually at right angles in the plane of a lamina and OZ is an axis perpendicular to the plane XOY of the lamina then $I_z = I_y + I_x$ where I_x, I_y and I_z are respectively the moments of inertia of the lamina about the axes OX, OY and OZ.

Let dm be the mass of a small element of the body at P, the co-ordinates of which are (x, y).

Let $OP = r$.

Then $I_x + I_y = \int y^2 \cdot dm + \int x^2 \cdot dm.$

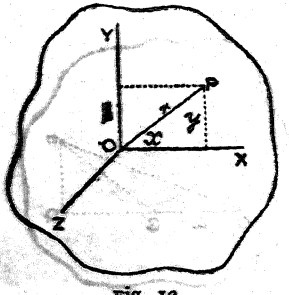


Fig. 12

$$= \int (x^2 + y^2)\, dm$$

$$= \int OP^2 \cdot dm = \int r^2 \cdot dm$$

But $\int r^2 \cdot dm$ is the moment of inertia of the body, I_z, about the axis OZ.

$\therefore I_z = I_x + I_y$.

27. Moment of inertia of a disc about a diameter

The moment of inertia of a circular disc about an axis passing through its centre 'O' and perpendicular to its plane $= \frac{Ma^2}{2}$.

If AB and CD are two diametres, which are mutually perpendicular, in the plane of the disc, the moment of inertia I of the disc about AB is equal to that about CD (since the distribution of the mass of the disc is similar about any diameter). From the theorem of perpendicular axes, we have

$$I + I = \frac{Ma^2}{2} \text{ or } I = \frac{Ma^2}{4}.$$

28. Moment of inertia of a cylinder about an axis passing through its centre, perpendicular to the axis of the cylinder

Let a cylinder AB be of length 'l', radius 'r' and mass per unit volume 'ρ'. The mass of the cylinder, M, is then equal to $\pi r^2 l \rho$.

Imagine the cylinder to be divided into thin circular discs perpendicular to the axis AB.

Consider one such disc, P, of thickness dx at a distance x from O, the midpoint of the cylinder.

Mass of the disc $= \pi r^2 \, dx. \rho$

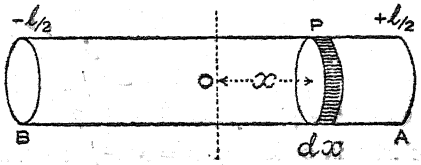


Fig. 13

Moment of inertia of the disc *about a diameter* which is parallel to the axis of rotation passing through O

$$= \frac{\pi r^2 \rho dx . r^2}{4}.$$

Moment of inertia of this disc about the axis of rotation passing through O $= \frac{\pi r^4 \rho dx}{4} + (\pi r^2 \rho dx) x^2$ (from Parallel Axes Theorem.)

Moment of inertia of the cylinder

$$= \int_{-l/2}^{+l/2} \frac{\pi r^4 \rho dx}{4} + \int_{-l/2}^{+l/2} \pi r^2 \rho x^2 dx$$

$$= \left[\frac{\pi r^4 \rho l}{4} + \frac{\pi r^2 \rho l^3}{12}\right]$$

$$= \frac{Mr^2}{4} + \frac{Ml^2}{12} = M\left(\frac{r^2}{4} + \frac{l^2}{12}\right)$$

29. It is evident that the moment of inertia of a cylinder, when the axis of rotation coincides with its axis, is equal to $\frac{Mr^2}{2}$.

30. Moment of inertia of a rectangular plate of sides 'a' and 'b'

(a) *About an axis through its centre, parallel to the side, a*

Let ρ be the mass per unit area of the plate. Imagine the rectangular plate to be divided into thin strips parallel to the side '*a*'.

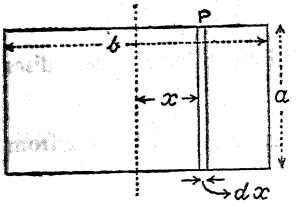


Fig. 14

Consider a strip at P of width dx and at a distance x from the axis.

Mass of the strip $= a.\ dx.\ \rho$.

Moment of inertia of the strip about the axis $= a.\ \rho.\ x^2 dx$.

Moment of inertia of the plate about the axis

$$\int_{-b/2}^{+b/2} a\rho x^2 dx = \frac{a\rho b^3}{12}.$$

If M is the mass of the plate, $M = a.\ b.\ \rho$.

$\therefore$ Moment of inertia of the plate $= Mb^2/12$ .... (1)

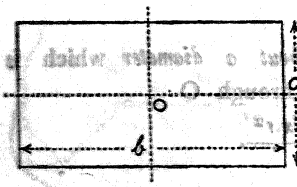


Fig. 15

(b) Similarly, it can be shown that the Moment of inertia of the plate about an axis passing through its centre but parallel to the side 'b' $= Ma^2 / 12$. (2)

(c) Hence from the theorem of perpendicular axes, the moment of inertia of the plate about an axis passing through its centre, perpendicular to its plane.

$$= \frac{Ma^2}{12} + \frac{Mb^2}{12} = M\frac{(a^2 + b^2)}{12}.$$

It may be noted that this expression $I = M.\frac{(a^2 + b^2)}{12}$ also gives the moment of inertia of a thick rectangular plate or of a bar, about an axis passing through its centre, and perpendicular to the face, the sides of which are a and b.

31. The values of the moments of inertia of some regular bodies are given below:

	Body		Axis of rotation	Moment of inertia
1.	Thin uniform rod (length l)	(a)	Through its centre and perpendicular to its length.	$\frac{Ml^2}{12}$
	,, ,,	(b)	Through one end and perpendicular to its length.	$\frac{Ml^2}{3}$
2.	Thin rectangular lamina (sides a and b)	(a)	Through its centre and parallel to a side 'a'	$\frac{Mb^2}{12}$
	,, ,,	(b)	Through its centre and parallel to a side 'b'.	$\frac{Ma^2}{12}$
	,, ,,	(c)	Through the centre of the plate perpendicular to the surface.	$M\left(\frac{a^2 + b^2}{12}\right)$
3.	Circular disc	(a)	Through its centre and perpendicular to its plane.	$\frac{MR^2}{2}$
	,, ,,	(b)	About a diameter	$\frac{MR^2}{4}$
4.	Solid cylinder (length l and radius of cross-section R)	(a)	Through its axis of rotation.	$\frac{MR^2}{2}$
	,, ,,	(b)	Through its centre and perpendicular to the length.	$M\left(\frac{l^2}{12} + \frac{R^2}{4}\right)$
5.	Solid sphere (radius R)		About a diameter.	$\frac{2}{5}MR^2$

32. Determination of a Fly-wheel

A fly-wheel is a massive wheel W with a long cylindrical axle A passing through its centre, perpendicular to its plane. The axle is mounted horizontally on ball bearings at its ends. There is a stout pin P fixed on the axle.

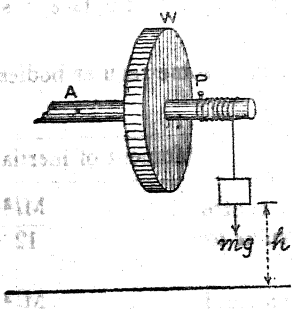


Fig. 16

A loop is made at one end of a light string and a convenient weight of mass m attached at the other end. The loop is slipped over the pin and the wheel is turned slowly so that the string winds itself round the axle without over-lapping. If the wheel is now let free, it begins to rotate in the opposite direction under the action of the descending weight and the string begins to unwind. The length of the string is adjusted in the beginning so that *the loop slips off the pin just when the weight, in its descent, touches the ground.* A mark is made on the wheel against some reference line to indicate its position when the loop slips off (i.e., just when the mass touches the ground).

The wheel is next turned so that the string winds itself on the axle a known number of times (say n_1). *Care should be taken to see that the turns of the string do not fall on one another.* Holding the wheel in this position, the height 'h' of the bottom of the weight above the floor is measured with a scale. The wheel is then let free so as to rotate under the action of the weight and a *stop-clock is started simultaneously.* The mass now descends and touches the ground after a time when the loop slips off the pin.

The time (t_1 *sec.*) taken by the string to unwind itself is noted from the stop-clock *without stopping it.* The wheel continues to rotate after the weight is detached until all the kinetic energy acquired by it is lost. The number of revolutions (n_2) made by the wheel before it comes to rest is noted. The stop-clock is stopped when the fly-wheel stops and the time taken (t_2 *sec.*) for the wheel to come to rest after the mass is detached, is calculated.

As the mass at the end of the string descends, it loses potential energy. But this loss in potential energy is equal to the kinetic

energy gained by the falling mass and the rotating wheel. A portion of the potential energy lost is also used up in the work done in overcoming friction.

The mass m falls through a height, h, and therefore the loss of potential energy of the mass $= mgh$ ... ... (1)

Let ω be the angular velocity of the wheel when the weight is detached and I be the moment of inertia of the wheel.

The kinetic energy of the wheel $= \frac{1}{2} I \omega^2$ .... (2)

If v be the velocity of the descending mass *just* before it touches the ground, the kinetic energy of the mass *just* before it touches the ground $= \frac{1}{2} mv^2$.

If r is the radius of the axle, the linear velocity of any point on the axle $= r\omega$.

As the axle rotates, the string unwinds itself and the velocity of the descending mass is equal to the velocity of the point on the axle, or $v = r\omega$.

$\therefore$ The kinetic energy of the mass $= \frac{1}{2} m\omega^2 r^2$ ... (3)

Let 'f' be the work done in overcoming the frictional resistance for one rotation of the wheel. The wheel turns through n_1 revolutions before the weight is detached. Hence the work done in overcoming friction in these n_1 revolutions $= n_1 f$ ... ... (4)

From the principle of conservation of energy, we have therefore

$$mgh = \tfrac{1}{2} I\omega^2 + \tfrac{1}{2} mr^2\omega^2 + n_1 f \text{ or}$$

$$2mgh = I\omega^2 + mr^2\omega^2 + 2n_1 f \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

The fly-wheel has taken t_1 sec. for the first n_1 revolutions performed before the weight is detached and t_2 sec. for the n_2 revolutions performed after the weight is detached before it comes to rest. During the first interval its angular velocity has increased from zero to ω, while in the second interval it has decreased from ω to zero. If we assume that the fly-wheel is subjected to a uniform frictional force, the average angular velocity becomes equal to $\frac{\omega}{2}$ in either interval.

$$\text{Hence } \frac{\omega}{2} = \frac{2\pi n_1}{t_1} = \frac{2\pi n_2}{t_2} \text{ or } \omega = \frac{4\pi n_1}{t_1} \quad \ldots \quad (6)$$

$$= \frac{4\pi n_2}{t_2} \quad \ldots\ldots \quad (7)$$

$$= \frac{4\pi (n_1 + n_2)}{(t_1 + t_2)} \quad \ldots\ldots \quad (8)$$

The angular velocity ω may be calculated by using either of the relations (6), (7) or (8).

After the mass has dropped down, the fly-wheel loses all its kinetic energy in rotating through n_2 revolutions, doing work against the frictional resistance.

$$\therefore \ \tfrac{1}{2} I\omega^2 = n_2 f. \quad \text{or} \quad f = \tfrac{1}{2} I \frac{\omega^2}{n_2}. \quad \ldots\ldots \quad (9)$$

Substituting for f in (5) we get

$$2mgh = I\omega^2 + mr^2\omega^2 + n_1 I \frac{\omega^2}{n_2}$$

$$2mgh - mr^2\omega^2 = I\omega^2 \left(1 + \frac{n_1}{n_2}\right)$$

$$\therefore \ I = \frac{2mgh - mr^2\omega^2}{\omega^2 (1 + n_1/n_2)} = \frac{n_2 m}{n_1 + n_2} \left(\frac{2gh}{\omega^2} - r^2\right) \quad \ldots\ldots \quad (10)$$

Thus determining ω, n_1, n_2, h and r as described above the moment of inertia of the fly-wheel may be calculated.

B - UNIFORM CIRCULAR MOTION

33. *A body moving along a circular path is said to be in circular motion. If its speed is constant, it is said to be in uniform circular motion.*

Consider a particle moving with a speed v along an arc of a circle with centre 'O' and radius equal to 'a' (Fig. 17).

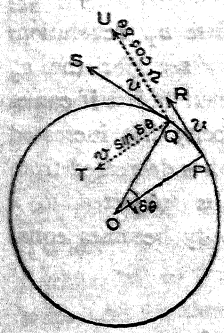


Fig. 17

Let the particle move from P to Q in a short interval of time δt and let $\angle POQ = \delta\theta$.

The velocity at P is v in the direction of the tangent PR and the velocity at Q is also v but in the direction of the tangent QS.

To determine the acceleration of the particle :

Resolve the velocities as P and Q parallel to PO and perpendicular to PO respectively.

1. Parallel to PO :

the component at P $= 0$

the component at Q $= v \sin \delta\theta = v.\ \delta\theta$ (since $\delta\theta$ is small)

2. Perpendicular to PO :

The component at P $= v$.

,, ,, Q $= v \cdot \cos \delta\theta = v$

(since $\delta\theta$ is very small)

$\therefore$ Acceleration parallel to PO

$$= \frac{v \cdot \delta\theta - O}{\delta t}$$

$$= \frac{v \cdot \delta\theta}{\delta t} = v \cdot \omega \text{ (} \omega \text{ being its angular velocity)}$$

$$= \omega^2 r = \frac{v^2}{r} \text{ (since } v = \omega \cdot r \text{).}$$

Acceleration, perpendicular to PO

$$\text{(i. e., along its tangent PR)} = \frac{(v - v)}{\delta t} = 0.$$

Thus the acceleration of the particle is wholly in the direction PO (i.e.,) it is normal to its direction of motion and its value is equal to $v\,\omega$ or $\omega^2 r$ or $\frac{v^2}{r}$. Hence *a particle moving along an arc of a circle is subject to an acceleration equal to* $\frac{v^2}{r}$ *in a direction normal to the direction of its motion.* It therefore follows that when a particle is having uniform circular motion it is acted on by a force. This force on the body acts towards the centre of the circle and its value is equal to $\frac{mv^2}{r}$ or $m\omega^2 r$ where m is the mass of the particle.

This force acting on the particle towards the centre of the circle is called the **centripetal force.**

34. According to Newton's third law of motion, there is a reaction to every force, which is equal and opposite to it. This force of reaction on a body in uniform circular motion therefore acts away from the centre of the circle and is equal to $\frac{mv^2}{r}$ (or $m\omega^2 r$). This force is called the **centrifugal force.**

It may be noted that the centrifugal force is caused by the inertia of the body and when the body is in circular motion, it (the centrifugal force) tends to take the body away from the centre of the circle, and if

there be no external force acting on the body, the body would move away from the centre of the circle due to the **centrifugal force.**

A particle moving in a circular path describes 2π radians about the centre in one complete revolution. Therefore the period of revolution,

$$T = \frac{2\pi}{\omega} \text{ and } \omega = \frac{2\pi}{T}.$$

There are many interesting applications of centrifugal force such as the centrifuge, the centrifugal pump, the governor of a steam engine etc.

Examples

1. *A uniform square lamina of side 2a is hung by one corner so that it can freely swing in its own plane. Find the moment of inertia of the plate about the axis of rotation.*

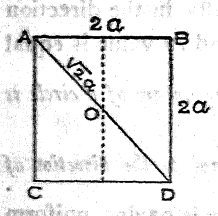


Fig. 18

Moment of inertia of the square lamina about an axis passing through its midpoint 'O' and perpendicular to the plane,

$$I_o = m\frac{(2a)^2+(2a)^2}{12} = \frac{2}{3}ma^2.$$

Let the plate be hung from the corner A.

M. I. about a parallel axis passing through

$$A = I_o + m\,.\,(OA)^2 = \tfrac{2}{3}ma^2 + m(\sqrt{2}\,.\,a)^2 = \tfrac{8}{3}ma^2.$$

2. *A particle of mass 2 lb. is being whirled in a horizontal circle at the end of a string of length 5 ft., the other end of the string being fixed. Find the greatest number of revolutions per minute the particle can make without breaking the string, if the maximum tension the string can bear is 150 lb. wt.*

Tension in the string = Centrifugal force acting on the mass.

Let n be the greatest number of revolutions per sec.

$\therefore$ Maximum value of angular velocity of rotation, $\omega = 2\pi n$.

Radius of the Circle, $r = 5$ft.

Mass of the body $m = 2$ lb.

Maximum tension in the string $= 4\pi^2 . n^2 . r . m = 4\pi^2 . n^2 . 5 . 2$

$= 40\pi^2 n^2$ poundals

But the maximum tension = 150 lb. wt.

$= 150 \times 32$ poundals.

Hence $40\,\pi^2 n^2 = 150 \times 32$

$$\text{or } n^2 = \frac{120}{\pi^2} = 12{\cdot}16$$

and $n = 3{\cdot}487$

∴ Maximum number of revolutions per minute = 209·2.

3. *A fly-wheel of mass 100 kg. and radius of gyration 50 cm. makes 120 revolutions per minute. Find the moment of inertia of the fly-wheel and the energy stored in it.*

Moment of inertia, $I = Mk^2$

Radius of gyration k = 50 cm.

Mass of the wheel = 100 kg.

Moment of inertia $= 100{,}000 \times 50^2$

$= 25 \times 10^7$ gm. cm.^{2}

No. of revolutions per sec., $n = 120/60 = 2$

Angular velocity, $\omega = 2\pi n = 2\pi \times 2 = 4\pi$

∴ K.E. stored in the fly-wheel $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 25 \times 10^7 \times (4\pi)^2$

$= 19{\cdot}73 \times 10^9$ ergs.

4. *A light string is wound 3 times round the axle (of radius 1·4 cm.) of a fly-wheel and a weight of 100 gm. is attached at the free end of the string. The weight is then released. The wheel is found to make 5 revolutions in 30 sec. before coming to rest after the mass is released. Find the M. I. of the fly-wheel.*

No. of revolutions, the wheel makes before the mass falls, $n_1 = 3$.

No. of revolutions, the wheel makes after the mass falls, $n_2 = 5$.

Mass attached, $m = 100$ gm.

Radius of the axle, $r = 1{\cdot}4$ cm.

Height through which the mass falls $= 2\pi r n_1$

$= 2 \times \frac{22}{7} \times 1{\cdot}4 \times 3 = 26{\cdot}4$ cm.

Average angular velocity just after the mass falls $= \frac{2\pi \times 5}{30} = \frac{\pi}{3}$

Angular velocity just when the mass falls, $\omega = \frac{2\pi}{3}$.

$$\text{M. I. of the fly-wheel} = \frac{n_2 m}{n_1 + n_2}\left(\frac{2gh}{\omega^2} - r^2\right) \text{ (vide p. 40)}$$

$$= \frac{5 \times 100}{3+5}\left(\frac{2 \times 980 \times 26{\cdot}4}{(2\pi/3)^2} - 1{\cdot}4^2\right)$$

$$= 73{\cdot}9 \times 10^5 \text{ gm.cm.}^2.$$

Exercises

1. Explain what you understand by moment of inertia and radius of gyration.

2. State and explain (a) theorem of parallel axes and (b) theorem of perpendicular axes.

3. Describe an experiment to determine the moment of inertia of a fly-wheel.

4. Obtain an expression for the normal acceleration of a body moving round a circular path with uniform speed.

5. A light string, wound 5 times round the axle of a fly-wheel carries at its free end a mass of 250 gm. When the system is released, the mass is detached after moving down a distance of 39·6 cm. Find the moment of inertia of the fly-wheel if it makes 8 revolutions in 72 sec. after the mass is released.

(*Ans.* $6{\cdot}056 \times 10^6$ gm.cm.^{2})

6. Show that the moment of inertia (1) of a circular disc of radius 'a' about a tangent line is $\frac{5ma^2}{4}$ and (2) of a solid sphere of radius 'a' about a tangent line is $\frac{7ma^2}{5}$.

7. A fly-wheel of mass 200 kg. and of radius 60 cm. makes 240 revolutions per minute. Assuming that the mass is concentrated at its rim, find the energy stored in the fly-wheel.

(*Ans.* $22{\cdot}74 \times 10^3$ joules)

8. A fly-wheel of mass 60 kg. and of radius of gyration 15 cm. is mounted on a horizontal axle of radius 2 cm. of negligible mass. A light string wound on the axle carries at its free end a mass of 3 kg., the mass hanging down freely. Assuming that there is no friction at the bearings show that the acceleration of the mass is $g/1126$ cm./sec.^{2}.

9. An electron of mass $0{\cdot}911 \times 10^{-30}$ kg. under the action of a magnetic force moves in a circle of radius 3 cm. at a speed of 3×10^6 metres/sec. At what speed will a proton of mass $1{\cdot}673 \times 10^{-27}$ kg. move in a circle of the same radius if it is acted on by the same magnetic force. [Ans. 7×10^4 metres/sec.]

10. Four masses, each of 5 lb. are fastened together with four strings each 3 ft. long, so as to form a square. If the square rotates in a horizontal plane on a smooth table at a speed of 2 revolutions per second find the tension in the strings. [Ans. 37 lb. wt.

11. A particle of mass 1 lb. is being whirled in a horizontal circle at the end of a string 4 ft. long, the other end being fixed. If the maximum tension the string can bear is 120 lb. wt., find the greatest speed per minute which can be given to the particle. [Ans. 41·84 rev./min.]

12. Define (*a*) moment of inertia and (*b*) radius of gyration.

Deduce expressions for (i) angular momentum and torque. Prove mathematically that when two spheres, one solid and the other hollow, of the same mass and radius are allowed to roll down an inclined plane from rest, it is the solid sphere that moves faster than the other. (A. U. April 1962)

13. Derive an expression for the centripetal force acting on a particle having uniform circular motion.

(A. U. Sep., 1962)

Chapter III

SIMPLE HARMONIC MOTION

35. Simple Harmonic Motion: Let P be a point moving in a circle with centre 'O' and radius 'a' with a uniform speed v. Let A'OA be a diameter of the circle and let PN be drawn perpendicular to A'OA. As P moves round the circle, the foot of the perpendicular N moves backwards and forwards on A'OA between A and A'. The motion of N is said to be **simple harmonic.** Let us investigate the motion of N.

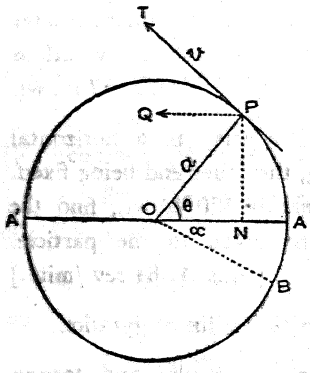


Fig. 19

36. Displacement of N: If ω be the angular velocity with which the radius vector OP rotates (and it may be called the angular velocity of P about O), $v = a\omega$.

If the displacement ON be called x and if $\angle$ AOP $= \theta$,

$$x = \text{OP} \cos\theta = a\cos\theta.$$

The position of P and consequently of N varies with time. If time is measured from the instant when P is at A,

$$\theta = \omega t \text{ and } x = a\cos\omega t.$$

If, however, time is measured from any arbitrary instant, say, when P is at B, so that $\angle$ BOP $= \omega t$, we have

$$\theta = \angle \text{BOP} - \angle \text{AOB} = \omega t - \angle \text{AOB}.$$

Now since B is a fixed point on the circle, $\angle$ AOB is a constant for the motion of P and if it be called α,

$$\theta = \omega t - \alpha \text{ and } x = a\cos(\omega t - \alpha) \quad \ldots \quad (1)$$

x gives the position of the particle having simple harmonic motion. For the given motion of P, the radius of the circle, a, is a constant. 'a' is called the **amplitude** of the simple harmonic motion.

32. Period of S. H. M.: Since ω is the angular velocity of P, it takes $\frac{2\pi}{\omega}$ sec. to describe one revolution. This is the period of revolution, T, of the point P. In the time P moves from P to A′ (in Fig. 19) on the circle, N moves from N to A′; in the time P moves from A′ to A on the circle, N moves from A′ to A and in the time P moves from A to P on the circle and completes the revolution, N moves from A to N (in Fig. 19) and thus as P completes one revolution, N will be passing the same position in the same direction. This motion of N between two successive instants when it is passing the same position in the same direction is called one **oscillation** and the interval of time between the two instants, i.e., the time taken for one oscillation, T, the **period** of the simple harmonic motion. It is seen that it (i.e., period) is the time taken by the particle describing simple harmonic motion for one complete backward and forward motion on the diameter of the circle AOA′.

Now this period, T, is equal to the period of revolution of P, which in turn is equal to $\frac{2\pi}{\omega}$.

$\therefore$ The period of S. H. M., $T = \frac{2\pi}{\omega}$.

It is seen that the period does not depend upon the amplitude 'a'; it depends only on the angular velocity, ω.

The number of oscillations performed per second is called the **frequency of oscillation**, n.

Obviously, $n = \frac{1}{T}$.

38. Velocity of N: The velocity of P is v in the direction PT (Fig. 19). The velocity of N which moves on AOA′ as P moves on the circle, is equal to the component of the velocity of P in the direction parallel to AOA′, i.e., $v \cos \angle TPQ$

(PQ being drawn parallel to AOA′).

But $\angle TPQ = \frac{\pi}{2} - \angle OPQ$

$= \frac{\pi}{2} - \angle PON$

∴ The velocity of N = The component of the velocity of P in the direction PQ.

$$= v \cos\left[\frac{\pi}{2} - \angle \text{PON}\right]$$

$$= v \sin \angle \text{PON} = v \sin \theta$$

$$= a\,\omega \sin \theta$$

$$= \omega\sqrt{a^2 - x^2} \quad \ldots \quad (2)$$

$$(\text{since } a \sin \theta = \text{PN} = \sqrt{a^2 - x^2})$$

39. Acceleration of N: The acceleration of N is equal to the component of the acceleration of P in the direction of the motion of N, i.e., of AOA'. (Fig. 19)

The acceleration of P $= \frac{v^2}{a} = \omega^2 a$ towards 'O'.

∴ The acceleration of N $= \frac{v^2}{a} \cos\theta$, towards 'O'

or $= \omega^2 a \cos\theta$, towards 'O'

$= \omega^2 . \text{ON}$, ,,

$= \omega^2 x$. ,, ... (3)

The acceleration of the particle executing simple harmonic motion is thus directed towards the centre and is proportional to its distance from the centre.

Since the acceleration is towards O, it is negative when N is to the right of O, i.e., when x is positive and, it is positive when x is negative. When we allow for the sign, therefore, the acceleration of N $= -\omega^2 x$.

It may be noted that ω^2 is a constant (since ω is a constant) and this constant determines the period of the simple harmonic motion in accordance with the relation, $T = \frac{2\pi}{\omega}$. *The period is independent of the amplitude; it depends only on the angular velocity, ω.*

40. Conditions for S. H. M.: Now it can be shown that the converse of the relation expressed in each of the equations, (1), (2) and (3) is also true; i.e., if any of these conditions is satisfied by a particle moving along a line, the particle has simple harmonic

motion. Of these three conditions, however, the last one relating to acceleration is preferred and S. H. M. is usually defined as follows :

A particle is said to have simple harmonic motion if its acceleration is directed towards a fixed point in its path and is proportional to distance (measured along the path) from the fixed point.

41. According to the conditions stated, the motion of the particle will be simple harmonic, whethr the path of the particle is straight or curved. If the path, AOA′, is curved as in Fig. 20 (*a*), imagine that it is straightened out as in Fig. 20 (*b*).

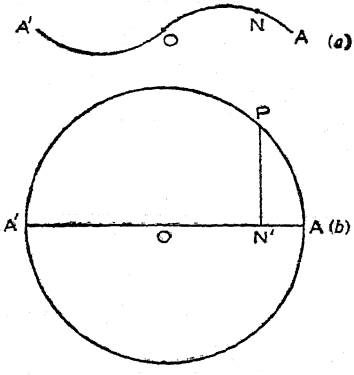


Fig. 20 (a) & (b)

Let N be a particle and let its acceleration be μ.ON, (where μ is a constant) and let it be directed towards O. Suppose A is the extreme end of the path. With O as centre and OA as radius, describe a circle. Now let P move on the circle with uniform angular velocity equal to $\sqrt{\mu}$.

Draw PN′ perpendicular to the diameter A′OA. Then the foot of the perpendicular N′ moves backwards and forwards along AOA′ as P moves on the circle and its acceleration is equal to μ.ON′ and is directed towards O. Let the motion of P on the circle be such that P is at A when N (the given particle) is at A. Then the foot of the perpendicular N′ is also at A at this instant. Now the particle N and the foot of the perpendicular N′ are both at the same point A at this instant and their velocities are zero (since their direction of motion is reversed at A).

Their accelerations which are equal to μ.ON and μ.ON′ are also equal at this instant since they coincide in position. Therefore, as N′ and N move along AOA′ they continue to coincide with each other and have the same displacement, velocity and acceleration. The motion of N is therefore identical with that of N′. Since the motion of N′ is simple harmonic, the motion of N also is simple harmonic and its period is equal to that of N′, i. e., $\frac{2\pi}{\sqrt{\mu}}$.

The period of N is independent of the amplitude of the simple harmonic motion. Such a motion is said to be **isochronous.** When a simple pendulum or a tuning fork vibrates, its periodic time remains constant though the extent of swing, i.e., the amplitude, gradually decreases. (These are the cases of isochronous motion.)

42. Equation of motion of a body in S. H. M.: Let P be a particle which moves on AB with its acceleration directed towards a point O on AB and proportional to its distance from O. When P is at a distance x from O, its acceleration is $\frac{d^2x}{dt^2}$. ... ... (1)

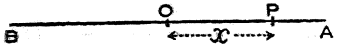


Fig. 21

Then $\frac{d^2x}{dt^2} \propto x$

or $\frac{d^2x}{dt^2} = -\mu x$ ... (2)

where μ is constant. The negative sign indicates that the acceleration is towards O.

From (2) we have

$$\frac{d^2x}{dt^2} + \mu x = 0. \quad \ldots \quad \ldots \quad (3)$$

When the motion of a particle satisfies this condition, the particle has S. H. M. and its period $= \frac{2\pi}{\sqrt{\mu}}$.

43. Force acting on a particle in S. H. M.

If the mass of the particle be m, the force acting on the particle

$$= m. \frac{d^2x}{dt^2}.$$

Multiplying (2) by m, we have

$$m. \frac{d^2x}{dt^2} = -m. \mu. x.$$

$$= -f. x. \quad \ldots \quad \ldots \quad (4)$$

where f is a constant $= \mu. m$ or $\mu = \frac{f}{m}$. ... (5)

It is thus seen that *if the force acting on a particle is proportional to its displacement from a fixed point O in its path and is directed towards O, the motion is simple harmonic* and the period T, is given by the relation

$$T = \frac{2\pi}{\sqrt{\mu}} = 2\pi / \sqrt{f / m} = 2\pi \sqrt{\frac{m}{f}}. \quad \ldots \text{ from (5)}$$

Now, when $x = 1$, m. $\frac{d^2x}{dt^2} = -f$. 'f' is thus equal to the force acting on the particle for unit displacement, and the negative sign indicates that the force acts towards O. 'f' may therefore be called the **restoring force per unit displacement.** Writing F for the force acting on the particle $\left(= m\frac{d^2x}{dt^2} \right)$, we have, if $F = -fx$, the motion to be simple harmonic and $T = 2\pi\sqrt{\frac{m}{f}}$ where m is the mass of the particle.

44. Work done in S. H. M.

The force acting on a particle, when its displacement is x from O $= f.x$.

The magnitude of this force may be assumed to be unaltered during a further small displacement, dx.

The work done upon the particle to displace it by $dx = fx.\ dx$.

∴ The work done in displacing particle from O through a distance $x = \int_0^x fx.\ dx. = f\left[\frac{x^2}{2}\right]_0^x = \frac{1}{2}fx^2$.

The work done on the particle is stored up as potential energy and it is thus seen that *the potential energy of a particle having S. H. M. is proportional to the square of its displacement from the mean position.*

Conversely, if the work done on a particle is proportional to the square of its displacement from a fixed point in its path (and is equal to kx^2), the motion of the particle is simple harmonic and it is seen from art. 43 that the period of oscillation $T = 2\pi\sqrt{\frac{m}{f}} = 2\pi\sqrt{\frac{m}{2k}}$ where m is the mass of the particle, since $f = 2k$.

45. Angular simple harmonic motion

A body rotating about an axis under the action of a couple may also be in simple harmonic motion. The corresponding equation for its motion can be shown to be given by $\frac{d^2\theta}{dt^2} + \mu\theta = 0$.

When this equation is satisfied by a body rotating about an axis, its motion is simple harmonic and its period of oscillation $= \frac{2\pi}{\sqrt{\mu}}$.

46. Phase of S. H. M.

The displacement, velocity and acceleration of a particle executing S. H. M. vary from instant to instant as the particle performs an oscillation. The particle is therefore said to have a certain **phase** at a particular instant. The phase changes continuously as the particle executes the oscillation and it repeats itself at intervals equal to the period. The particle, therefore, passes through the same series of phases in every oscillation.

In the expression $x = a \cos(\omega t - \alpha)$, the angle $(\omega t - \alpha)$ is taken to define the phase of the S. H. M. at the instant of time represented by t. If there are two S. A. M.s of the same period given by the relations $x_1 = a_1 \cos(\omega t - \alpha_1)$

and $x_2 = a_2 \cos(\omega t - \alpha_2)$,

the two S. H. M.s differ in phase by $(\omega t - \alpha_1) - (\omega t - \alpha_2) = (\alpha_2 - \alpha_1)$ which is a constant. The two S.H.M.s will therefore have a constant differnce of phase.

If $(\alpha_2 - \alpha_1) = 0$, the two S. H. M.s are said to *agree in phase*, and if $(\alpha_2 - \alpha_1) = \pi$, they are said to be *opposite in phase*.

If, however, the periods are different and the S.H.M.s are given by the relations,

$$x_1 = a_1 \cos(\omega_1 t - \alpha_1)$$

and $x_2 = a_2 \cos(\omega_2 t - \alpha_2)$, the difference of phase between them $= (\omega_1 t - \alpha_1) - (\omega_2 t - \alpha_2) = (\omega_1 - \omega_2)t - (\alpha_1 - \alpha_2)$ and it varies with time.

47. Simple Pendulum

An ideal simple pendulum consists of a *point mass* suspended from a fixed point by means of a *light inextensible* string. The point mass is called the **bob** of the pendulum and it oscillates in a vertical plane under the action of gravity. The motion of a simple pendulum is *simple harmonic* as is shown below.

Consider a simple pendulum of length l suspended from a point O. If the bob (of mass m) is drawn to a side and released, the pendulum begins to oscillate in the vertical plane. Let A and C represent the extreme positions of the bob and B, the equilibrium position.

Let OC make an angle θ with the vertical. The force acting on the bob at C $= mg$, vertically downwards. Resolving it along OC and perpendicular to OC, we get the components to be (1) $mg \cos \theta$ along OC and (2) $mg \sin \theta$ perpendicular to OC i.e., along the tangent to the arc ABC and it is directed towards the equilibrium position B.

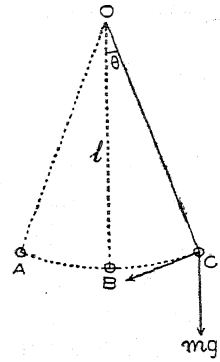


Fig. 22

But if θ is small, $\sin \theta = \theta$ and therefore the force acting on the bob towards the equilibrium position, $B = mg \cdot \theta = mg \cdot \frac{BC}{l}$.

$\therefore$ The acceleration of the particle towards $B = \frac{g}{l} \cdot BC = \frac{g}{l} \times$ (displacement of the bob from B), i.e., acceleration $\propto$ displacement of the bob from B ($\because$ g and l are constant). This holds good for any position of the bob in its path AC.

The motion of the particle is therefore simple harmonic and its period of oscillation,

$$T = \frac{2\pi}{\sqrt{\mu}} = 2\pi \Big/ \sqrt{\frac{g}{l}} = 2\pi \cdot \sqrt{\frac{l}{g}}.$$

Thus for small oscillations (i.e., when θ is small) the oscillations of a simple pendulum are isochronous and the period is given by the relation, $T = 2\pi \sqrt{\frac{l}{g}}$.

48. An expression for the period of the simple pendulum may also be deduced by considering its energy.

The energy of the pendulum at the extreme position is entirely potential, while at the equilibrium position B, it is entirely

kinetic. But at any other point, the pendulum possesses energy which is partly kinetic and partly potential. *From the principle of conservation of energy, the total energy of the pendulum at any instant in its path is constant.*

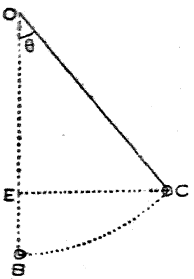


Fig. 23

Consider the pendulum in any position OC as it oscillates, when the string makes an angle θ with the vertical.

If l be the length of the pendulum and m the mass of the bob, the potential energy of the bob in the position C (with reference to B)

$$= mg \times BE$$
$$= mg\,(l - l\cos\theta) = mgl\,(1 - \cos\theta). \quad \ldots\ldots \quad \ldots\ldots \quad (1)$$

The angular velocity of the bob in this position $= \frac{d\theta}{dt}$.

$\therefore$ The corresponding linear velocity of the bob $= l \cdot \frac{d\theta}{dt}$.

$\therefore$ Kinetic energy of the bob $= \frac{1}{2} ml^2 \left(\frac{d\theta}{dt}\right)^2$. $\ldots\ldots$ (2)

Hence for the total energy of the bob, we have

$$\frac{1}{2} ml^2 \left(\frac{d\theta}{dt}\right)^2 + mgl\,(1 - \cos\theta) = \text{constant}. \quad \ldots\ldots \quad (3)$$

Differentiating equation (3) with respect to time, we get

$$\frac{1}{2} ml^2 \cdot 2\frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} + mgl \sin\theta \frac{d\theta}{dt} = 0.$$

i. e., $$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin\theta = 0$$

or $$\frac{d^2\theta}{dt^2} + \frac{g}{l} \cdot \theta = 0 \text{ (when } \theta \text{ is small)}.$$

This is an equation for a body in angular simple harmonic motion.

Hence the motion of the simple pendulum is simple harmonic and its period, $T = 2\pi \Big/ \sqrt{\frac{g}{l}} = 2\pi \sqrt{\frac{l}{g}}$ $\ldots\ldots$ (4)

Using the relation (4), the acceleration due to gravity, 'g', at any place may be determined.

49. The Compound Pendulum

A simple pendulum is an ideal one and it cannot be achieved in practice because *it is impossible to obtain a point mass* and also because *the string cannot be weightless.* These factors make it necessary that we should take into consideration the moment of inertia of the whole system about the point of suspension. In addition, *the string slackens as the bob approaches the extreme positions of its swing on either side* (Fig. 24). *It would therefore be preferable to take a rigid body and make it oscillate in a vertical plane about a horizontal axis.* A rigid body oscillating in this way is called a **compound pendulum.**

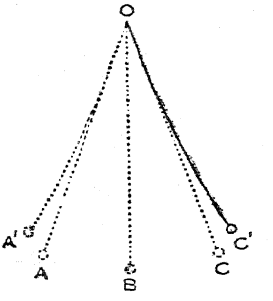


Fig. 24

50. An expression for its period of oscillation may be deduced as follows :

Consider a rigid body of mass m suspended from a horizonta knife-edge passing through a point 'O' in it.

Let G be its centre of gravity. The weight of the pendulum acts vertically downwards through G. When the pendulum is at rest, G will lie vertically below O. Let $OG = l$. If the body be now displaced through an angle and released, it begins to oscillate about the knife-edge passing through 'O'.

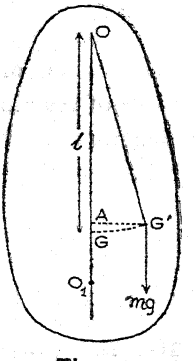


Fig. 25

Let G' be the position of the centre of gravity G at any instant and let $\angle GOG' = \theta$.

Draw G' A perpendicular to OG. Then AG is the vertical height through which the centre of gravity is raised, so that the potential energy gained by the body $= mg.$ AG.

$$= mg\,(OG - OA) = mg\,(l - l\cos\theta) = mgl\,(1 - \cos\theta) \quad \ldots \quad (1)$$

The angular velocity ω of the pendulum in this position $= \frac{d\theta}{dt}$. If I is the moment of inertia of the body about 'O', the kinetic energy of the pendulum $= \frac{1}{2} I \omega^2 = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2$ .... (2)

But *the sum of the two energies must be constant* according to the principle of the conservation of energy.

$$\therefore \tfrac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 + mgl\,(1-\cos\theta) = \text{constant.} \quad \ldots \quad (3)$$

Differentiating equation (3) with respect to time, we get

$$\tfrac{1}{2} \cdot I \cdot 2 \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} + mgl \sin\theta \frac{d\theta}{dt} = 0,$$

$$\text{i.e., } I \frac{d^2\theta}{dt^2} + mgl \sin\theta = 0.$$

Hence, if θ is small, $\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0.$ .... (4)

The motion of the body is, therefore, simple harmonic and its period of oscillation,

$$T = 2\pi \Big/ \sqrt{\left(\frac{mgl}{I}\right)} = 2\pi \sqrt{\frac{I}{mgl}} \quad \ldots \quad \ldots \quad (5)$$

[**Alternatively**

The weight of the body 'mg' acting through its centre of gravity 'G' in the displaced position gives rise to an equal and opposite force of reaction at the knife-edge 'O' (Fig. 25). The two together constitute a couple which tends to restore the pendulum to its equilibrium position.

Moment of the couple $= mg \times G'A.$

But $G'A = l \sin\theta = l\theta$ (when θ is small).

$\therefore$ Moment of the couple $= mgl\theta.$

If I is the moment of inertia of the body about 'O', angular acceleration $= \frac{mgl}{I} \cdot \theta$. i.e., angular acceleration $\propto \theta$.

$\therefore$ The motion of the compound pendulum is simple harmonic and its period T is given by $T = 2\pi \Big/ \sqrt{\frac{mgl}{I}} = 2\pi \sqrt{\frac{I}{mgl}}$]

If, now, I_g be the moment of inertia of the body about a parallel axis passing through G, we have $I = I_g + ml^2$.

If k is the radius of gyration of the body about G,

$I_g = mk^2$ so that $I = mk^2 + ml^2 = m(k^2 + l^2)$.

Substituting for I in equation (5), we get

$$T = 2\pi \sqrt{\frac{m(k^2 + l^2)}{mgl}} = 2\pi \sqrt{\frac{k^2 + l^2}{gl}} \quad \ldots \quad (6)$$

Hence the period of a compound pendulum is the same as that of a simple pendulum of length

$$\frac{k^2 + l^2}{l} = \left(\frac{k^2}{l} + l\right) \quad \ldots \quad (7)$$

$\left(\frac{k^2 + l^2}{l}\right)$ is called the **length of the equivalent simple pendulum** (L) and it is always greater than l since k^2 is always positive.

51. Centres of suspension and oscillation

Take a point O_1 on OG produced so that $OO_1 \frac{(k^2+l^2)}{l} = L$.

Then O is called the **centre of suspension** and O_1, the **centre of oscillation.** (Fig. 25)

If $GO_1 = l_1$, $OO_1 = l + l_1$.

$\therefore \frac{k^2 + l^2}{l} = l + l_1$ or $\frac{k^2}{l} + l = l + l_1$

or $\frac{k^2}{l} = l_1$. i.e., $k^2 = ll_1$.

$$\therefore T = 2\pi \sqrt{\frac{k^2 + l^2}{lg}} = 2\pi \sqrt{\frac{l_1 + l}{g}} = 2\pi \sqrt{\frac{L}{g}} \ldots (1)$$

If the pendulum is now reversed and suspended from a horizontal knife-edge passing through O_1 (the centre of oscillation), the period of oscillation,

$$T_1 = 2\pi \sqrt{\frac{k^2 + l_1^2}{l_1 g}} = 2\pi \sqrt{\frac{ll_1 + l_1^2}{l_1 g}}$$

$$= 2\pi \sqrt{\frac{l + l_1}{g}} = 2\pi \sqrt{\frac{L}{g}} \quad \ldots \quad (2)$$

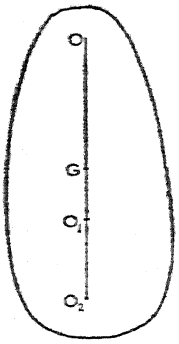


Fig. 26

Hence the period of oscillation is the same, whether the body is suspended from O or O_1. The centres of oscillation and suspension are thus **interchangeable** and the period of oscillation about either of them is equal to the period of a simple pendulum of length L $(=l+l_1)$, the length between the two points about which the periods are equal.

It may be seen from considerations of symmetry that if a point O_2 be chosen on OG produced such that $GO_2 = GO$, the period of oscillation about O_2 will also be equal to T (Fig. 26). It is to be noted that the length of the simple equivalent pendulum is not that between O and O_2 but that between O and O_1. It is therefore the length between two points which are **asymmetric** with respect to the centre of gravity of the body and about which the periods are the same.

(That there are two values of l, for which the period T is the same, is evident from the relation, $T = 2\pi \sqrt{\frac{k^2+l^2}{lg}}$ which reduces to the quadratic equation in l,

$$l^2 - \frac{gT^2}{4\pi^2} l + k^2 = 0.$$

Similarly, there will be two more points on the other side of G having the same period.

Thus, there will be four points having the same period, T, which are collinear with the centre of gravity—two of them at a distance l and the other two at a distance l_1 on either side of G).

51. (a) Condition for the period to be minimum

The period of the compound pendulum T becomes minimum when $k = l$ as shown below:

$$T = 2\pi \sqrt{\frac{k^2 + l^2}{lg}} \quad \text{(Eq. 6. — Art. 50)}$$

$$\text{or} \quad T^2 = 4\pi^2 \left(\frac{k^2 + l^2}{lg} \right) = \frac{4\pi^2}{g} \left(\frac{k^2}{l} + l \right)$$

Differentiating with respect to l, we get

$$2T \frac{dT}{dl} = \frac{4\pi^2}{g} \left\{ -\frac{k^2}{l^2} + 1 \right\} = \frac{4\pi^2}{g} \left(1 - \frac{k^2}{l^2} \right)$$

When T is minimum, $\frac{dT}{dl} = 0$

$$\therefore \frac{4\pi^2}{g}\left(1 - \frac{k^2}{l^2}\right) = 0$$

$$\text{or } \frac{k^2}{l^2} = 1 \quad \text{or } k^2 = l^2$$

i. e., $k = \pm l$.

52. Determination of 'g' with a compound pendulum

A compound pendulum is usually in the form of a bar AB, which is provided with a number of holes along its length. The bar is made to vibrate in a vertical plane about a horizontal knife-edge, which can be passed through any one of the holes. The holes are usually symmetrical about the centre of gravity of the bar.

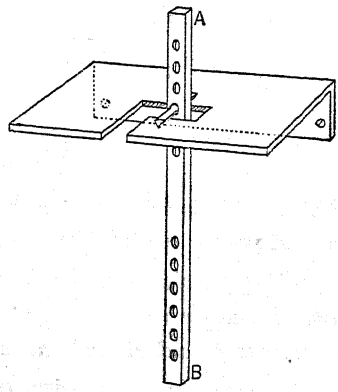


Fig. 27

To begin with, the horizontal knife-edge is passed through the hole near the end A. The period of oscillation is determined and the distance of the knife-edge from the end A is measured. The experiment is repeated changing the position of the knife-edge from hole to hole. In each case the period and the *distance of the position of the knife-edge from the same end A are noted.* When the knife-edge gets near the centre of gravity of the bar, the period becomes very long. The bar is then inverted and the experiment is repeated on the other side of the bar changing the knife-edge from hole to hole as before. In each case the period, T, is determined and the *distance from the same end A of the bar* is measured.

A graph is plotted between the period (on y—axis) and the distance from A (on the x—axis). Two symmetrical curves ABC and DEF are obtained, one for either side of the centre of gravity (Fig. 28). It is seen that, as the distance from A increases, the period first decreases and then increases tending to infinity as the centre of gravity of the bar is approached.

A horizontal line PQRS is drawn cutting both the curves at points P, Q, R and S. P, Q, R and S are then the four points on

the bar collinear with the centre of gravity having the same period.

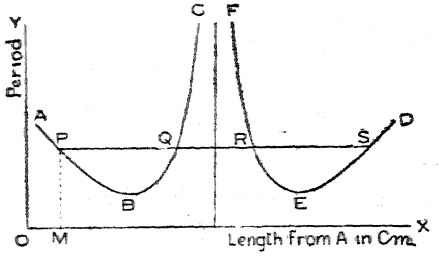


Fig. 28

Now PR (=QS) gives the length of the equivalent simple pendulum having a period equal to PM (=T). PR and QS are measured and their average is taken as the length, L, of the equivalent simple pendulum.

The acceleration due to gravity 'g' is then calculated from the relation, $T = 2\pi\sqrt{\frac{L}{g}}$ or $g = 4\pi^2\frac{L}{T^2}$.

To be more accurate, the average value of g is obtained by drawing some more lines parallel to the $x-$axis and finding the corresponding values of L and T.

53. Kater's pendulum

The principle of the interchangeability of the centres of oscillation and suspension was employed by Captain Kater in 1817, to determine the length of the seconds pendulum and the acceleration due to gravity 'g' very accurately.

The pendulum consists of a bar having two knife-edges A and B near its ends. It carries two pairs of cylinders C and F and D and E,

Fig. 29

each pair having the same size. C and F are larger and D and E, much smaller. Also C and D are heavy and are usually made of brass and E and F, are of wood. The knife-edges and the cylinders are movable along the bar and they are to be fixed in positions so that the pendulum is symmetrical about its centre of figure. The centre of gravity of the pendulum will then be to one side and the knife-edges will be asymmetric with respect to the centre of gravity.

The periods of oscillation are determined suspending the bar from the two knife-edges in turn. The positions of the weights are then adjusted so that the periods about the two knife-edges are equal.

Kater, in his experiments, succeeded after a long trial in getting a negligible difference in the periodic time

about the two knife-edges A and B. It may be of interest to note that the number of oscillations made by his pendulum in 24 hours about the two knife-edges respectively *did not differ* by more than a fraction of a swing.

A and B then give the reciprocal points of oscillation and suspension and the distance between them is the length, L, of the equivalent simple pendulum. The length between the knife-edges A and B is carefully measured and 'g' calculated from the relation, $g = \frac{4\pi^2 L}{T^2}$.

The experiment is however subject to some **sources of error** which require correction. They are due to (*1*) *yielding of the support* (*2*) *buoyancy and viscosity of air* (*3*) *curvature of the knife-edges and* (*4*) *finite arc of swing.* One serious difficulty in conducting the experiment is that a very long time extending over days may have to be spent for securing perfect equality of periods.

54. Graphical Method

When the periods are nearly equal in Kater's pendulum, the displacemet of any knife-edge towards the centre of gravity of the pendulum shortens the periodic time about the knife-edge. Now the length of the equivalent simple pendulum is obtained as follows :

The knife-edges A and B are fixed symmetrically with respect to the bar near the ends. The periods T_1 and T_2 about the two knife-edges A and B respectively are determined. The length l between the knife-edges is measured. Let the period T_1 about A be greater than the period T_2 about B (i.e., $T_1 > T_2$).

The knife-edges are now moved equally by a few centimetres towards the centre of the bar and fixed. The periods T_1' and T_2' respectively about the two knife-edges A and B are again determined. T_1' should be less than T_2' (i. e., $T_1' < T_2'$).

Otherwise : If, however, $T_1' > T_1$ and $T_2' < T_2$, the knife-edges are moved further apart by equal distances and the periods about the knife-edges determined so that the condition, $T_1' < T_2'$ is satisfied.

The length l' between the knife-edges A and B is measured.

A graph is plotted between the periods on the y—axis and the lengths (between the knife-edges) on the x—axis choosing a large scale for the purpose. The points K and N correspond to the periods T_1 (about A) and T_2 (about B) when the length between the knife-edges is l and the points M and L correspond to the periods T_2' (about B) and T_1' (about A) respectively when the length between the knife-edges is l'.

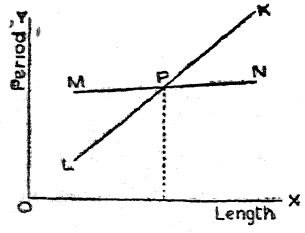


Fig. 30

KL and MN are then joined. The co-ordinates of P, the point of their inter-section, then give the length L of the equivalent simple pendulum and its corresponding period, T. 'g' may then be calculated from the relation,

$$g = \frac{4\pi^2 L}{T^2}.$$

5. Bessel's modified formula

Bessel showed that *it was not necessary to adjust the periods about the two knife-edges to be exactly equal* as Kater attempted to do in his experiments. Bessel suggested a modified formula for 'g' when the periods about the two knife-edges are *nearly equal.*

Let T_1 and T_2 be the two periods obtained about the two knife-edges respectively. Let l_1 and l_2 be the distances of the two knife-edges from the centre of gravity of the pendulum.

$$\text{Now } T_1 = 2\pi\sqrt{\frac{k^2 + l_1^2}{l_1 g}} \text{ or } T_1^2 = 4\pi^2 \frac{k^2 + l_1^2}{l_1 g}$$

$$\text{i.e., } T_1^2 l_1 = \frac{4\pi^2}{g}(k^2 + l_1^2) \quad \ldots \quad \ldots \quad (1)$$

$$\text{Similarly, } T_2 = 2\pi\sqrt{\frac{k^2 + l_2^2}{l_2 g}} \text{ or } T_2^2 = 4\pi^2 \left(\frac{k^2 + l_2^2}{l_2 g}\right)$$

$$\text{or } T_2^2 l_2 = \frac{4\pi^2}{g}(k^2 + l_2^2) \quad \ldots \quad \ldots \quad (2)$$

$$\text{From (1) and (2), } (T_1^2 l_1 - T_2^2 l_2) = \frac{4\pi^2}{g}(l_1^2 - l_2^2) \ldots (3)$$

(a) If $T_1 = T_2 = T$, this gives

$$T^2 (l_2 - l_2) = \frac{4\pi^2}{g}(l_1^2 - l_2^2)$$

and if l_1 is not equal to l_2, we will have

$$T^2 = \frac{4\pi^2}{g}(l_1 + l_2), = 4\pi^2 \frac{L}{g},$$

a relation already deduced.

(b) If T_1 is slightly different from T_2,

let $T_1^2 = T^2 + t^2$ ... ... (4)

and $T_2^2 = T^2 - t^2$ ... . (5)

Substituting for T_1^2 and T_2^2 in (3) we get

$$(T^2 + t^2) l_1 - (T^2 - t^2) l_2 = \frac{4\pi^2}{g} (l_1^2 - l_2^2)$$

$$\text{or } T^2 (l_1 - l_2) + t^2 (l_1 + l_2) = \frac{4\pi^2}{g}(l_1^2 - l_2^2)$$

$$\text{i.e., } \frac{T^2}{(l_1+l_2)} + \frac{t^2}{(l_1-l_2)} = \frac{4\pi^2}{g} \quad .... \quad .... \quad (6)$$

But from (4) and (5) $T^2 = \frac{T_1^2+T_2^2}{2}$ and $t^2 = \frac{T_1^2-T_2^2}{2}$

so that equation (6) can be written as $\frac{T_1^2+T_2^2}{2(l_1+l_2)} + \frac{T_1^2-T_2^2}{2(l_1-l_2)} = \frac{4\pi^2}{g}$.

$(l_1 + l_2)$ is the distance between the two knife-edges and it is determined very accurately with a cathetometer. l_1 and l_2 are next determined, finding the centre of gravity of the bar, by balancing it on a knife-edge, and measuring the distances of the knife-edges from the centre of gravity. As $(T_1^2 - T_2^2)$ is a very small quantity (since T_1 and T_2 are very nearly equal), any small error that might be made in measuring l_1 and l_2 in this manner will not introduce more than a *negligible error* in the result.

56. Method of coincidences

The method of coincidences is used to measure the period of a pendulum by comparing its oscillations with those of another of known period. This method is applicable in the case of two pendulums of nearly equal periodic times and also when the period of one is nearly a sub-multiple of the other.

The experimental pendulum A is placed in front of a **standard seconds pendulum B.** A small white disc, with a black line drawn vertically on it, is attached to the bob of the seconds pendulum B. A thin tail-piece, having preferably the same width as the line on the disc, is fixed to the end of the pendulum A. The vertical line and the tail-piece are conveniently observed through a telescope. The pendulum B and the telescope are so arranged that the line on B is seen through the telescope only when the pendulum B is passing through the equilibrium position. The pendulum A is adjusted so that when it is in the equilibrium position, the tail-piece hides the black line.

Suppose that *the two pendulums are set off from their equilibrium positions simultaneously.* If the pendulum A has a slightly longer period than B, then, B returns to the equilibrium position earlier and the vertical line on it can be seen through the telescope as it passes across. B gains over A to a greater extent at the next transit and still more in the next swing and so on. This continues until B gains half a swing over A. The two pendulums then cross their equilibrium positions simultaneously in opposite directions, and the vertical line on B is hidden by tail-piece on A. *Observation is continued until the two pendulums pass their equilibrium positions in the same direction simultaneously, which is indicated by the tail-piece hiding the line on B. A coincidence then takes place and in this interval of time, B has gained one complete oscillation over A.*

If the pendulum B makes n oscillations between two consecutive coincidences, A would have made only $(n-1)$ oscillations in the same interval. Since the period of the seconds pendulum B is 2 sec. the interval between the two coincidences is equal to $2n$ sec. and the periodic time of A is $\left[\frac{2n}{(n-1)}\right]$.

If A on the other hand gains over B, then the periodic time of A $= \left[\frac{2n}{(n+1)}\right]$, n being the number of oscillations made by B between two consecutive coincidences.

Examples

1. *A test tube float of weight 10 gm. and of external diameter 2 cm. is floated vertically in water by loading it with 15 gm. of mercury at its bottom. The tube is set in vertical oscillations. Find its period.*

Total mass of the float = 10 + 15 = 25 gm.

External radius of the tube = 1 cm.

Let the tube be depressed through a small distance, x at any instant in its oscillations.

Weight of water displaced

$$= \pi r^2 x \cdot d \cdot g = \pi \times 1^2 \times x \times 1 \times g = \pi x g \text{ dynes.}$$

Upward thrust acting on the tube $= \pi x g$ dynes.

Hence acceleration of the tube or the restoring force per unit displacement, $\mu x = \frac{\pi x g}{25} = \frac{\pi g}{25} \cdot x$ or $\mu = \frac{\pi g}{25}$.

$$\therefore \quad T = 2\pi \sqrt{\frac{1}{\mu}} = 2\pi \sqrt{\frac{25}{\pi g}} = 0.566 \text{ sec.}$$

2. *A particle is executing S. H. M. in a straight line. If its velocities are 25 cm./sec. and 16 cm./sec. respectively when it is at 2 cm. and 3 cm. from its position of equilibrium, find (a) the period of oscillation (b) the maximum velocity and (c) the amplitude.*

We know that velocity v of a body in S. H. M. in a straight line is given by $v = \omega \sqrt{a^2 - x^2}$,

(ω being the angular velocity, a the amplitude and x the distance from the centre.)

Thus $25 = \omega \sqrt{a^2 - 4}$

and $16 = \omega \sqrt{a^2 - 9}$

$$\therefore \quad \frac{25}{16} = \sqrt{\frac{a^2 - 4}{a^2 - 9}} \quad \text{or} \quad 25^2 (a^2 - 9) = 16^2 (a^2 - 4)$$

i.e., $(625a^2 - 256a^2) = (625 \times 9) - (256 \times 4)$.

Hence $a^2 = 12.48$ and $a = 3.532$ cm.

$$\text{Angular velocity, } \omega = \frac{25}{\sqrt{a^2 - 4}} = \frac{25}{\sqrt{12.48 - 4}} = 8.884 \text{ rad./sec.}$$

$$\text{Period of oscillation, } T = \frac{2\pi}{\omega} = \frac{2\pi}{8.814} = 0.7127 \text{ sec.}$$

$$\text{Maximum velocity} = a \cdot \omega = 3.532 \times 8.814 = 31.13 \text{ cm./sec.}$$

3. *A uniform square lamina of side 2 l is hung by one corner so as t swing freely in a vertical plane. Find the length of the equivalent simpl pendulum.*

M. I. of a rectangular lamina of sides 'a' and 'b' about ar axis passing through its centre and perpendicular to the plane $= m \cdot \frac{a^2 + b^2}{12}$.

M. I. of a square lamina of side $2\ l$ about a similar axis. $I_o = m \cdot \frac{(2l)^2 + (2l)^2}{12} = \frac{2}{3} ml^2$. When the lamina is hung from one corner, the axis about which the lamina swings is an axis parallel to the above and is at a distance $x = \sqrt{2} \cdot l$.

M. I. about a parallel axis, I

$$= I_o + mx^2 = \frac{2}{3} \cdot ml^2 + 2ml^2 = \frac{8ml^2}{3}.$$

Length of equivalent simple pendulum $= \frac{I}{m \cdot x} = \frac{4\sqrt{2}}{3} \cdot l.$

4. *A particle of weight Mg hangs at the end of a vertical elastic string of unstretched length L cm. and modulus of elasticity* λ *(i.e., Young's modulus). (a) Prove that the vertical oscillations of the system are simple harmonic and (b) find an expression for the period of its oscillation.*

Let OP be the string of length L and let it be stretched to Q when the mass is suspended at its lower end. Let the extension, PQ $= l$.

Fig. 31

$$\text{The linear strain} = \frac{\text{Increase in length}}{\text{Original length}} = \frac{l}{L}.$$

$$\therefore \text{ Stress acting on the elastic string} = \lambda \cdot \frac{l}{L}.$$

But the force acting on the string
= Stress × area of cross-section

$$= \lambda \cdot \frac{l}{L} \cdot A \text{ (where A is the area of cross-section of the string)} \quad \ldots \quad \ldots \quad (1)$$

Now that the string is in equilibrium

$$\lambda \cdot \frac{l}{L} \cdot A = Mg \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

(assuming the string to be of negligible mass).

If the mass be now pulled downwards and released, it will execute oscillations up and down.

Consider the forces acting on the mass when it is displaced by an amount x below Q.

Since the total extension of the string is equal to $(l + x)$, force acting upon the string in the upward direction due to its elasticity $= \lambda \cdot \frac{A}{L} (l + x)$.

Force acting downwards $= Mg$.

Resultant upward force on the mass

$$= \lambda \frac{A}{L} (l + x) - Mg$$

$$= \left[\lambda \cdot \frac{A}{L} (l + x) - \lambda \cdot \frac{A}{L} \cdot l \right] \ldots \text{ from (2)}$$

$$= \frac{\lambda A}{L} \cdot x \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$\therefore$ The acceleration of the mass $= \left(\frac{\lambda A}{LM} \right) \cdot x = \mu x \quad \ldots \quad (4)$

and it is directed towards Q.

Since the acceleration of M is proportional to its distance x from Q and is also directed towards Q, the motion of the system is simple harmonic and the period of oscillation, $T = \frac{2\pi}{\sqrt{\mu}} = 2\pi \sqrt{\frac{LM}{\lambda A}} \ldots (5)$

[If the mass of the string is not negligible, it can be shown that $T = 2\pi \sqrt{\frac{L}{\lambda A} (M + S/3)}$ S being the mass of the string].

But from (2) $\frac{LM}{\lambda A} = \frac{l}{g}$

$\therefore T = 2\pi \sqrt{\frac{l}{g}}$.

(l being the length through which the string is stretched by the mass M).

Thus the period of oscillation is the same as that of a simple pendulum of length l.

(To determine λ, the mass suspended at the end of the string is increased by a known amount m and the corresponding increase in length 'a' measured.

Then $\lambda = \left(\frac{mg}{A}\right)\left(\frac{L}{a}\right)$, L being the original length of the string before stretching.)

Exercises

1. Define S. H. M.

2. Find the velocity, acceleration and periodic time of a particle executing simple harmonic motion.

3. Define frequency and phase of S. H. M.

4. Show that the motion of a simple pendulum is simple harmonic. Hence deduce the period of oscillation of the simple pendulum.

5. Define 'centre of suspension' and 'centre of oscillation'. Show that they are interchangeable in a compound pendulum.

6. Give the theory of Kater's pendulum and explain how it can be used to determine 'g'.

7. Obtain an expression for 'g' in terms of two nearly equal periods of oscillation obtained about the two knife-edges of a Kater's pendulum.

8. Describe an accurate method of determining the period of oscillation of a pendulum.

9. Show that there are four points collinear with the *c.g.* of a compound pendulum bar having the same period, T.

10. Show that the period of vibration of a magnet suspended so as to rotate freely in a horizontal plane is equal to $2\pi\sqrt{\frac{I}{MH}}$.

11. A particle of mass m is moving with S. H. M. in a straight line. The particle is found to possess velocities v_1 and v_2 when it is at distances s_1 and s_2 from its position of equilibrium. Show that its period is $2\pi\sqrt{\frac{s_2^2 - s_1^2}{v_1^2 - v_2^2}}$.

12. A test tube float of mass 16 gm. is set in vertical oscillations on the surface of water. If its external diameter is 2 cm. find its period of oscillation. (*Ans.* 0·453 sec.)

13. Calculate the distance between the centres of oscillation and suspension on a uniform cylindrical bar used as a seconds pendulum. (g = 978 cm. / sec.^{2}) (*Ans.* 99.1 cm.)

14. A circular disc of radius 'a' is arranged to oscillate freely about a horizontal axis which is (1) tangential to the disc and (2) perpendicular to the plane of the disc and passing through a point on its edge. Find the lengths of the simple equivalent pendulum in each case. [*Ans.* $5a/4$; $3a/2$]

15. The distance between the knife-edge of a Kater's pendulum is 94·5 cm. The times of oscillations about the two knife-edges are 1·924 and 1·940 sec. respectively. Determine the acceleration due to gravity at the place if the *c. g.* of the pendulum lies at a distance of 66·5 cm. from the knife-edge about which the period is 1·940 sec.
[*Ans.* 979·7 cm./sec.^{2}]

16. Derive the expression for the period of oscillation of a compound pendulum.

Explain how Kater's pendulum is used for determining the acceleration due to gravity at any given place. (A. U. Sept. 1961)

17. Define centre of suspension and centre of oscillation as applied to a compound pendulum, and prove that they are interchangeable.

Describe Kater's pendulum and explain how it is used to determine the acceleration due to gravity at a place.
(A. U. April, 1964)

18. Obtain a formula for the period of a compound pendulum. What is the length of the simple equivalent pendulum? Determine the condition for the period to be minimum. (A. U. April, 1965)

19. Describe with relevant theory, an accurate method of determining the acceleration due to gravity at a place. Mention briefly how the acceleration due to gravity varies over the earth's surface. (A. U. March, 1966)

Chapter IV

ELASTICITY

57. When a system of forces, which are in equilibrium, is applied to a body, it generally undergoes a deformation, i. e., a change of size or a change of shape or both. Some bodies regain their original shape or size when the applied forces are removed while others cannot do so. Bodies which regain their original shape and size are called **elastic bodies** and those that cannot regain them are said to be **plastic.** No body, in nature, is either perfectly elastic or perfectly plastic.

Elasticity is the name given to that property of matter in virtue of which a body resists any change of shape or size produced in it by the application of external forces and recovers from the change when the forces are removed.

58. Strain

The deformation in shape or size or both suffered by a body when a system of forces is applied on it is called strain. Strain may be of different kinds, depending upon the nature of the forces applied.

(a) **Bulk Strain:** An isotropic body, when it is compressed uniformly in all directions, will have a change in volume without any change in its shape. All bodies—solid or fluid—are capable of undergoing this strain. *The strain in these cases is measured by the ratio of the change in volume to the original volume of the body.* If a change of volume equal to v be produced in a body of volume, V, the bulk or volume strain $= \frac{v}{V}$.

(b) **Linear Strain:** When a rod or wire is acted on by a system of forces, parallel to its length, it will undergo a change in length. If the forces are tensile, the body is elongated and if they are compressive, it is shortened, in the direction of the forces. *The strain in this case is measured by the ratio of the increase (or decrease) in length to the original length of the rod or wire.*

Thus if a change in length, l, is produced in a rod or wire of length, L, the linear strain $= \frac{l}{L}$.

(c) **Shearing Strain or Shear**: In the above two cases, the direction in which the forces act is perpendicular to the surface of the body on which they act. If, however, forces are applied parallel to its surface, they tend to produce relative displacement between the different layers of the body and the body is deformed. The strain produced is called a **Shear.** In the case of a shear, it is found that there is no change of volume of the body but only a change of shape. Liquids and gases cannot resist forces, however small, which act tangential to their surfaces and they flow under the action of such forces. *Solids alone can have a shearing strain.*

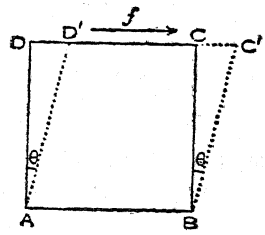


Fig. 32

The strain, shear, is measured by the ratio of the relative displacement between two layers to the distance between them. It is also given by the angle θ shown in Fig. 32. When θ is small, $\theta = \tan\theta = \frac{CC'}{BC}$

$$= \frac{\text{Relative displacement between the layers AB \& CD}}{\text{Distance between AB \& CD}}$$

59. Stress

When forces are applied to produce deformation in a body they are distributed over definite areas and further they form a system which is in equilibrium. Corresponding to this system of applied forces, equal and opposite forces are brought into play within the material of a body (AB) across any of its cross-sections

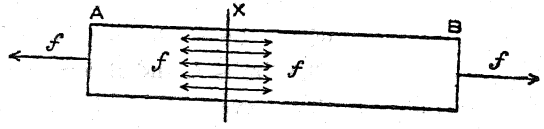


Fig. 33

(say at X) due to its elasticity. *These internal forces acting across any cross-section as at X within the body and which are in equilibrium between themselves constitute a* **stress.** It may be noted that *the force acting across any cross-section is equal to the force applied on the body.*

Quartz and phosphor bronze show practically no elastic after-effect and are therefore used widely for suspensions in measuring instruments like galvanometers.

63. Elastic fatigue

Consider two equal bodies A and B suspended by two metal wires of the same material, length and cross-section. When the bodies are turned through an angle and released, they perform torsional oscillations which are governed by the rigidity of the wire. Let the pendulum A be made to execute the oscillations for a fairly long time continuously, while B is at rest. Now if A and B be set to oscillate, with same amplitude to start with, it will be observed that A comes to rest sooner than B. This is attributed to '**elastic fatigue**' that is said to have been acquired by the torsion pendulum A on account of its previous continuous oscillations. If, however, A is given sufficient rest and made to oscillate again, it is observed that it comes to rest at the same time as the pendulum B, the fatigue having passed off.

64. Moduli of Elasticity

According to Hooke's law, stress $\propto$ strain, (when the strain is not too great,) i.e., $\frac{\text{Stress}}{\text{Strain}}$ = a constant.

This constant is called **modulus of elasticity** and is measured in dynes per sq. cm.

There are different moduli of elasticity depending upon different kinds of strain.

(a) **Young's modulus, Y** : Let a wire of length 'L' and of area of cross-section 'a' undergo an extension, l, when a stretching force, f, is applied in the direction of its length.

Then, longitudinal strain $= \frac{l}{\mathrm{L}}$

and longitudinal stress $= \frac{f}{a}$

The modulus of elasticity, in this case, is called **Young's modulus** and is usually denoted by the letter Y.

Hence $\mathrm{Y} = \frac{f/a}{l/\mathrm{L}} = \frac{f \cdot \mathrm{L}}{a \cdot l}$.

(b) **Bulk modulus,** k: When a body is subjected to a uniform pressure (which is compressive force per unit area), its volume decreases and the strain produced will be a change in the size only. The ratio of stress to strain in this case is called **bulk modulus** and it is represented by the letter, k.

If the diminution in the volume produced is v and the original volume is V, the strain $= \frac{v}{V}$.

If P is the pressure applied (i.e, the stress)

$$\text{bulk modulus, } k = \frac{\text{Stress}}{\text{Strain}} = \frac{P}{v/V} \text{ or } \frac{PV}{v}.$$

(c) **Rigidity modulus,** n: In the case of shearing strain, the ratio of the shearing stress applied to the face of the body to the shear produced, is called the **modulus of rigidity** and is usually denoted by the letter, n.

The shearing stress is measured by the tangential force per unit area, T and the strain is measured by the angle of shear, θ (in radians).

$$\text{Hence, modulus of rigidity, } n = \frac{\text{Stress}}{\text{Strain}} = \frac{T}{\theta}.$$

(d) **Poisson's Ratio,** σ: When a tensile stress is applied to a wire, the wire undergoes not only an extension of length in the direction of the force but also a lateral contraction.

Any measurement of length, taken in a direction perpendicular to the length of the wire, will have a diminution. *The ratio of the decrease in length in the lateral direction to the length in that direction is the lateral contraction.*

The ratio of lateral contraction to the longitudinal elongation is called **Poisson's ratio** and is denoted by the letter σ.

$$\text{Thus } \sigma = \frac{\text{lateral contraction}}{\text{longitudinal elongation}} \text{ and}$$

lateral contraction $= \sigma \times$ (longitudinal elongation).

65. Dimensions of modulus of elasticity

Strain is the ratio of two physical quantities which have the same dimensions. It is therefore a pure number and has zero

dimensions. When shear is measured as an angle also, it has zero dimensions since an angle has no dimensions. Since modulus of elasticity is the ratio of stress to strain and since strain has no dimensions, *the modulus of elasticity will have the same dimensions as those of stress.*

Stress is the ratio of force to area and, therefore, its dimensions are $\left[\frac{MLT^{-2}}{L^2}\right] = \left[ML^{-1}T^{-2}\right]$.

These, therefore, are also the dimensions of modulus of elasticity.

66. Shear and Linear strain

Let the face ABCD of a cube be distorted into ABC'D' by a system of tangential forces f applied as shown in Fig. 35. Then, θ, the angle through which BC or AD is turned, gives the shearing strain. It is obvious from Fig. 35 that the diagonal AC is elongated to AC' and the diagonal BD (at right angles to AC) is shortened to BD'. The elongation and contraction shown in the figure are much exaggerated. But in practice, the elongation and contraction and also the angle of shear, θ, are very small.

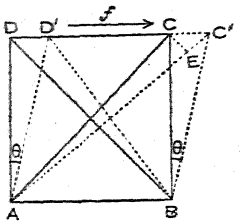


Fig. 35

Let CE be the perpendicular drawn from C on to AC'.

Then $\triangle$ CEC' may be considered to be a right angled isosceles triangle since θ is very small and $\angle$ CC'E does not differ much from 45°. Further AC may be considered to be equal to AE.

Now, the strain in the direction AC

$$= \frac{AC' - AC}{AC} = \frac{AC' - AE}{AC} = \frac{EC'}{AC}.$$

But $EC' = \frac{CC'}{\sqrt{2}}$ and $AC = \sqrt{2}\,BC$

$\therefore$ Strain (elongation) in the direction AC

$$= \frac{EC'}{AC} = \frac{CC'}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}\,BC} = \frac{1}{2}\frac{CC'}{BC} = \frac{1}{2}\theta$$

Similarly, the strain (contraction) in the direction of the diagonal BD may also be shown to be equal to $\frac{\theta}{2}$.

Thus, **an alternative view** may be taken of a shear. *When a body undergoes an elongation in one direction and an equal contraction in a perpendicular direction, the deformation produced is a shear. The elongation or contraction is numerically equal to half the angle of shear.*

67. We may also establish a relation between the stresses operating in these two alternative cases in the following way.

Consider a unit cube ABCD. Let the tangential forces T and T acting on the faces BA and DC produce a shearing strain. The two forces constitute a couple tending to tilt the body about B.

To keep the body in equilibrium, an equal and opposite couple is brought into play and let the components of the new couple be T and T but acting along DA and BC respectively. These forces are also equal to the stresses acting, since the cube is of unit side and the faces are of unit area. The shearing stress therefore is equal to T.

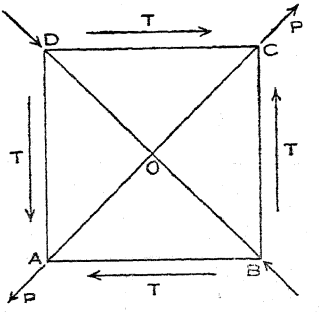


Fig. 36

Now resolve the forces T and T acting along BC and DC respectively in a direction perpendicular to the diagonal BD. The components are then each equal to T cos 45° $= \frac{T}{\sqrt{2}}$, they act in the same direction parallel to OC and are tensile.

∴ These components are together equivalent to a force $\frac{T}{\sqrt{2}} + \frac{T}{\sqrt{2}} = \sqrt{2}.T$ which acts normally across the surface represented by the diagonal BD.

Now, the area of the surface, BD $= \sqrt{2} \times 1 = \sqrt{2}$ (since the dimension perpendicular to the plane of the figure is 1 cm.).

∴ The normal force acting per unit area on the surface BD, i.e., the normal stress, $P = \frac{\sqrt{2}.T}{\sqrt{2}} = T$ or $P = T$.

Similarly, the compression stress acting on the plane AC can also be shown to be equal to T.

Corresponding to the normal stress P, there is a linear strain e in the direction OC, and it has been shown that $e = \frac{\theta}{2}$.

$$\therefore \text{ Rigidity modulus, } n = \frac{T}{\theta} = \frac{P}{2e}$$

68. Volume strain and Linear strain

Consider a unit cube subjected to a uniform dilatational stress. Let the linear elongation produced in each side be e.

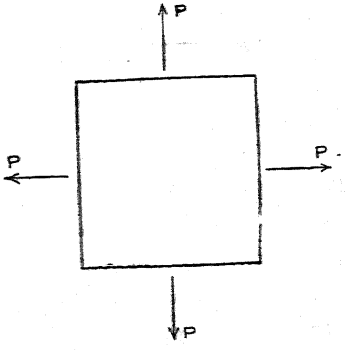


Fig. 37

Then the length of the side becomes $(1 + e.)$

$\therefore$ The volume of the cube

$= (1 + e)^3 = 1 + 3e + 3e^2 + e^3$

$= 1 + 3e$ approximately

($\because$ e is very small)

$\therefore$ Volume strain

$= 1 + 3e - 1 = 3e.$

If the volume (or bulk) strain produced is v, then $v = 3e$.

Hence the bulk strain is **thrice** the linear strain.

69. Relations between Y, n, k, and σ

Consider a unit cube ABCDA'B'C'D' the sides of which are parallel to one or another of the axes of reference OX, OY, and OZ respectively (Fig. 38).

Let normal tensile forces P, Q and R act on the faces of the cube in directions parallel to OX, OY and OZ respectively ;

i.e., the force P acts perpendicular to faces AA'D'D and BB'C'C,

the force Q acts perpendicular to faces AA'B'B and CC'D'D

and the force R acts perpendicular to faces ABCD and A'B'C'D'

respectively.

The forces P, Q and R are also the stresses since the areas of the faces over which they act respectively are of unit area.

The tensile stresses P, Q and R produce longitudinal elongations in their own directions and lateral contractions in the other two perpendicular directions respectively.

Now Young's modulus, $Y = \dfrac{\text{Longitudinal stress, P}}{\text{Longitudinal strain, } e}$

$\therefore$ Longitudinal strain, $e = \dfrac{\text{Longitudinal stress, P}}{Y}$

But lateral contraction $= \sigma$. (Longitudinal strain)

$= \sigma . e = \sigma . \dfrac{P}{Y}$, (Art. 64)

Thus a stress P acting along OX produces

a longitudinal elongation $= \dfrac{P}{Y}$ in the direction OX,

a lateral contraction $= \sigma \dfrac{P}{Y}$ in the direction OY

and a lateral contraction $= \sigma \dfrac{P}{Y}$ in the direction OZ.

Similarly, a stress Q acting along OY produces a longitudinal elongation $\dfrac{Q}{Y}$ in the direction OY and lateral contractions equal to $\left(\sigma \dfrac{Q}{R}\right)$ in each of the directions, OX and OZ.

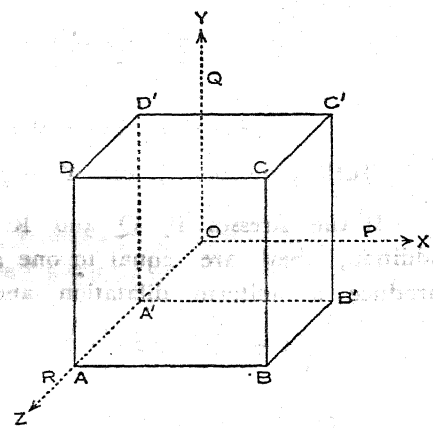


Fig. 38

And, a stress R acting along OZ causes a longitudinal elongation $\dfrac{R}{Y}$ in the direction OZ and lateral contractions equal to $\left(\sigma \dfrac{R}{Y}\right)$ in each of the directions OX and OY.

The strains produced by the action of P, Q and R may well be represented in a tabular form as follows:

Stress	Strain produced in the directions		
	OX	OY	OZ
P acting along OX	$\frac{P}{Y}$	$-\frac{\sigma P}{Y}$	$-\frac{\sigma P}{Y}$
Q acting along OY	$-\frac{\sigma Q}{Y}$	$\frac{Q}{Y}$	$-\frac{\sigma Q}{Y}$
R acting along OZ	$-\frac{\sigma R}{Y}$	$-\frac{\sigma R}{Y}$	$\frac{R}{Y}$
Stresses P, Q & R acting simultaneously	$\frac{P}{Y}-\frac{\sigma}{Y}(Q+R)$	$\frac{Q}{Y}-\frac{\sigma}{Y}(P+R)$	$\frac{R}{Y}-\frac{\sigma}{Y}(P+Q)$

(*—ve* sign indicates that it is a contraction)

(a) Bulk modulus, k and Young's modulus, Y

If the stresses P, Q and R act simultaneously and if in addition, they are equal to one another, i.e., P = Q = R, they produce a uniform dilatation and each side will be equal to

$$\left(1+\frac{P}{Y}-2\sigma\frac{P}{Y}\right)=1+\frac{P}{Y}(1-2\sigma).$$

$$\therefore \text{The linear strain} = \left[1+\frac{P}{Y}(1-2\sigma)-1\right]=\frac{P}{Y}(1-2\sigma)$$

and the bulk strain $= \frac{3P}{Y}(1-2\sigma)$ from Art. 68.

Thus if k is the bulk modulus of the material, k = stress, P/bulk strain,

$$\text{or bulk strain} = \frac{P}{k} \quad \ldots \quad \ldots \quad (1)$$

$$\text{i.e.,} \quad \frac{3P(1-2\sigma)}{Y}=\frac{P}{k} \quad \ldots \quad \ldots \quad (2)$$

$$\text{or} \quad Y=3k(1-2\sigma). \quad \ldots \quad \ldots \quad (3)$$

(b) **Modulus of rigidity, n and Young's modulus, Y**

If only the stresses P and Q act and if in addition, P is tensile and Q is compressive (i.e., P = − Q) the resulting strain is a shear. If n is the modulus of rigidity,

$$n = \frac{T}{\theta} = \frac{P}{2e} \text{ from Art. 67} \quad .... \quad .... \quad (4)$$

$$\text{or } e = \frac{P}{2n}. \quad .... \quad .... \quad .... \quad (5)$$

Now the linear strain, along the direction OX (when R = 0),

$$e = \frac{P}{Y} - \frac{\sigma}{Y}(Q) = \frac{P}{Y} - \frac{\sigma(-P)}{Y} \quad (\because P = -Q)$$

$$= \frac{P}{Y}(1 + \sigma) \quad ... \quad ... \quad (6)$$

Substituting for e in (5) $\quad \frac{P}{2n} = \frac{P}{Y}(1 + \sigma)$

$$\text{or } Y = 2n(1 + \sigma) \quad ... \quad ... \quad (7)$$

(c) **Other relations**

From (3) and (7), we have

$$Y = 3k(1 - 2\sigma) \text{ and } Y = 2n(1 + \sigma)$$

$$\therefore 3k(-2\sigma) = 2n(1 + \sigma).$$

$$\text{Hence } \sigma = \frac{3k - 2n}{2(3k + n)} \quad .... \quad .... \quad (8)$$

Eliminating σ in (3) and (7), we get

$$Y = \frac{9kn}{3k + n} \quad .... \quad .... \quad .... \quad (9)$$

70. Theoretical limits of σ

We have the equations $Y = 2n(1 + \sigma)$ .... (1)

and $Y = 3k(1 - 2\sigma)$ .... (2)

Since Y and n are positive, it follows from (1) that σ must be greater than − 1.

Since, in addtion, k is also positive, it is seen from (2) that σ cannot be greater than $\frac{1}{2}$.

Hence the theoretical limits for σ are − 1 and $\frac{1}{2}$.

Since σ also is positive, its value lies between 0 and $\frac{1}{2}$.

The strains produced by the action of P, Q and R may well be represented in a tabular form as follows:

Stress	Strain produced in the directions		
	OX	OY	OZ
P acting along OX	$\frac{P}{Y}$	$-\frac{\sigma P}{Y}$	$-\frac{\sigma P}{Y}$
Q acting along OY	$-\frac{\sigma Q}{Y}$	$\frac{Q}{Y}$	$-\frac{\sigma Q}{Y}$
R acting along OZ	$-\frac{\sigma R}{Y}$	$-\frac{\sigma R}{Y}$	$\frac{R}{Y}$
Stresses P, Q & R acting simultaneously	$\frac{P}{Y}-\frac{\sigma}{Y}(Q+R)$	$\frac{Q}{Y}-\frac{\sigma}{Y}(P+R)$	$\frac{R}{Y}-\frac{\sigma}{Y}(P+Q)$

(*—ve* sign indicates that it is a contraction)

(a) Bulk modulus, k and Young's modulus, Y

If the stresses P, Q and R act simultaneously and if in addition, they are equal to one another, i.e., P = Q = R, they produce a uniform dilatation and each side will be equal to

$$\left(1+\frac{P}{Y}-2\sigma\frac{P}{Y}\right)=1+\frac{P}{Y}(1-2\sigma).$$

$$\therefore \text{The linear strain} = \left[1+\frac{P}{Y}(1-2\sigma)-1\right]=\frac{P}{Y}(1-2\sigma)$$

and the bulk strain $= \frac{3P}{Y}(1-2\sigma)$ from Art. 68.

Thus if k is the bulk modulus of the material, k = stress, P/bulk strain,

$$\text{or bulk strain} = \frac{P}{k} \quad \ldots \quad \ldots \quad (1)$$

$$\text{i.e.,} \quad \frac{3P(1-2\sigma)}{Y}=\frac{P}{k} \quad \ldots \quad \ldots \quad (2)$$

$$\text{or } Y=3k(1-2\sigma). \quad \ldots \quad \ldots \quad (3)$$

(b) **Modulus of rigidity, n and Young's modulus, Y**

If only the stresses P and Q act and if in addition, P is tensile and Q is compressive (i.e., $P = -Q$) the resulting strain is a shear. If n is the modulus of rigidity,

$$n = \frac{T}{\theta} = \frac{P}{2e} \text{ from Art. 67} \quad .... \quad .... \quad (4)$$

$$\text{or } e = \frac{P}{2n}. \quad .... \quad .... \quad .... \quad (5)$$

Now the linear strain, along the direction OX (when $R = 0$),

$$e = \frac{P}{Y} - \frac{\sigma}{Y}(Q) = \frac{P}{Y} - \frac{\sigma(-P)}{Y} \quad (\because P = -Q)$$

$$= \frac{P}{Y}(1 + \sigma) \quad ... \quad ... \quad (6)$$

Substituting for e in (5) $\quad \frac{P}{2n} = \frac{P}{Y}(1 + \sigma)$

$$\text{or } Y = 2n(1 + \sigma) \quad ... \quad ... \quad (7)$$

(c) **Other relations**

From (3) and (7), we have

$$Y = 3k(1 - 2\sigma) \text{ and } Y = 2n(1 + \sigma)$$

$$\therefore 3k(-2\sigma) = 2n(1 + \sigma).$$

$$\text{Hence } \sigma = \frac{3k - 2n}{2(3k + n)} \quad .... \quad .... \quad (8)$$

Eliminating σ in (3) and (7), we get

$$Y = \frac{9kn}{3k + n} \quad .... \quad .... \quad .... \quad (9)$$

70. Theoretical limits of σ

We have the equations $Y = 2n(1 + \sigma)$ .... (1)

and $Y = 3k(1 - 2\sigma)$ .... (2)

Since Y and n are positive, it follows from (1) that σ must be greater than -1.

Since, in addtion, k is also positive, it is seen from (2) that σ cannot be greater than $\frac{1}{2}$.

Hence the theoretical limits for σ are -1 and $\frac{1}{2}$.

Since σ also is positive, its value lies between 0 and $\frac{1}{2}$.

71. Work done in producing strain

Work is done by an impressed force in producing strain in any body. The work spent in producing the strain is stored up in the body as potential energy.

It can be shown that the *work done per unit volume* in **producing strain** of any kind (linear or shear or bulk) is equal to [$\frac{1}{2}$ (**stress**) × (**strain**)].

(a) **Linear Strain:** Let a force f act on a wire of length l and of area of cross-section 'a' and extend it by x. Work done in stretching the wire by dx = force × distance = $f \times dx$.

$$\therefore \text{Work done for the whole extension} = \int_0^x f \times dx.$$

If Y is the Young's modulus of the material of the wire

$$\frac{f}{a} = \text{Y}. \frac{x}{l} \quad \text{or} \quad f = \text{Y}. \frac{ax}{l}.$$

$$\therefore \text{Work done} = \int_0^x \frac{\text{Y}ax}{l} \cdot dx = \frac{\text{Y}}{l} \cdot a \cdot \frac{x^2}{2}.$$

Now the volume of the wire $= a \,.\, l$.

$$\therefore \text{Work done per unit volume} = \frac{\text{Y}}{l} \cdot \frac{ax^2}{2} \cdot \frac{1}{a.l}$$

$$= \tfrac{1}{2}\left(\frac{\text{Y}ax}{l.a}\right)\left(\frac{x}{l}\right) = \tfrac{1}{2}(\text{stress}) \times (\text{strain}).$$

(b) **Shear Strain:** Let a tangential force f, acting in the direction DC of a cube ABCD, shear it through an angle θ.

Let the side BC = l and CC' = x.

Then the stress $= \frac{f}{l^2}$

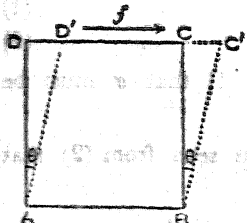


Fig. 38 (a)

and shear strain, $\theta = \frac{x}{l}$

$$\therefore n = \frac{f}{l^2.\theta} \quad \text{or} \quad f = n.l^2.\theta.$$

Substituting for θ, we have

$$f = n.l^2.\frac{x}{l} = n.x.l$$

and stress $= \frac{f}{l^2} = \frac{n.x.l}{l^2}$.

Work done to displace the layer AB by $dx = f.dx$.

$\therefore$ Work done for the displacement x =

$$\int_0^x f.dx = \int_0^x nlx.dx = nl\,\frac{x^2}{2}.$$

Hence, work done per unit volume

$$= \frac{\frac{1}{2}.n.l.x^2}{l^3} = \frac{1}{2}\left(\frac{nlx}{l^2}\right)\left(\frac{x}{l}\right)$$

$= \frac{1}{2}$ (stress) $\times$ (strain).

(c) **Bulk Strain :** Let a pressure P be applied uniformly all over a body of volume V and let v be the change in volume produced.

Now the stress applied = P and, strain $= \frac{v}{V}$.

Work done to produce a small change in volume dv = P . dv*

Work done for the whole change $= \int_0^v P.\,dv$

But $k = \frac{P}{v/V}$ or $P = \frac{kv}{V}$.

$\therefore$ Work done $= \int_0^v \frac{kv}{V}\,dv = \frac{kv^2}{2V}$

and work done per unit volume

$$= \frac{kv^2}{2V.V} = \frac{1}{2}\left(\frac{kv}{V}\right)\left(\frac{v}{V}\right)$$

$= \frac{1}{2}$ (stress) $\times$ (strain).

---

*Let e be the contraction produced along the edges of each one of a unit cube when a uniform pressure P is applied to the faces of the cube as in Fig. 38. Then the work done by the stress P in the direction of x—axis = P$\times e$.

Similarly, work done in the direction of y—axis = P$\times e$;
and work done in the direction of z—axis = P$\times e$.

$\therefore$ The total work done = P$\times 3e$

But $3e = dv$ (Art. 68)

$\therefore$ The total work done = P$\times dv$.

72. Determination of Young's modulus

Young's modulus of the material of a wire may be determined by directly measuring the extension of the wire, produced by a given load.

The apparatus used for this determination consists of two long wires of the same meterial suspended side by side from the same support. One of the wires, C, which is a comparison wire, carries a constant load at its lower end sufficient to keep it taut. The other, the experimental wire, E, carries a weight hanger, M, at its lower end on which weights can be put. The extension is measured by a scale, S and a vernier V. One of the wires, usually the comparison wire, carries the scale and the other, the vernier. By suspending the two wires from the same support, *the errors due to the yielding of the support and the change in temperature are eliminated.*

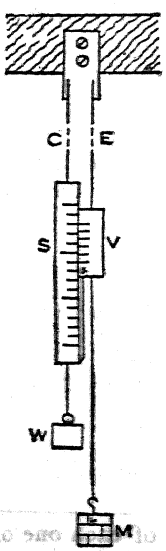


Fig. 39

A sufficient dead load (m) must be put on the hanger to keep the experimental wire also *taut.*

The extension of a wire, on the first loading or unloading after an interval of time, is not usually the same as that for subsequent loadings. *The wire will have to be loaded and unloaded a number of times to get it into a steady state.*

The wire is therefore loaded and unloaded a number of times in equal steps of 500 gm. up to the maximum load to be used in the experiment.

Starting with the dead load alone, the reading of the vernier on the scale is read. Then the load on the hanger is increased in steps of 500 gm. up to a maximum of 3·5 kg. and the corresponding readings of the vernier for each load are observed.

After the maximum load of 3·5 kg. is reached, the wire is unloaded by the same steps of 500 gm. each time and the corresponding readings of the vernier are noted. The average of the two readings for each load—one obtained while the load is increased and the other while the load is decreased—is taken and the extension for each load calculated.

The results are tabulated thus :

Serial No.	Load on the wire	Vernier reading		Mean	Extension	Extension for 2 kg. (M)
		Load increasing	Load decreasing			
1	m	a_o'	a_o''	a_o		
2	$m + 500$	a_1	a_2	a	$a - a_o$	
3	$m + 1000$	b_1	b_2	b	$b - a_o$	
4	$m + 1500$	c_1	c_2	c	$c - a_o$	
5	$m + 2000$	d_1	d_2	d	$d - a_o$	$d - a_o$
6	$m + 2500$	e_1	e_2	e	$e - a_o$	$e - a$
7	$m + 3000$	f_1	f_2	f	$f - a_o$	$f - b$
8	$m + 3500$	g_1	g_2	g	$g - a_o$	$g - c$

The extension (l) for the load, M ($= 2$ kg.) is calculated by obtaining the differences ($d - a_o$), ($e - a$), ($f - b$), ($g - c$) and taking their mean.

Removing the load entirely (excepting the dead load), the mean diameter ($2r$) and the length (L) of the experimental wire are measured.

Young's modulus of the material of the wire is then calculated from the relation

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{Mg / \pi r^2}{l / L} = \frac{MgL}{\pi r^2 l}.$$

A graph between load and extension is plotted. The graph will be a straight line in accordance with Hooke's law.

73. There are also several other alternative methods of measuring the extension.

(i) In the Searle's extensometer (Fig. 40) a spirit level in combination with a micrometer screw is used. One end of the spirit level is supported on a frame hanging from the comparison wire and the other end on the tip of a micrometer screw which is fitted to another frame suspended from the experimental wire. The

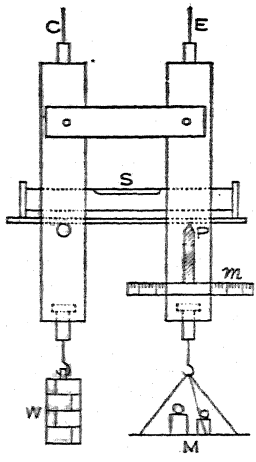


Fig. 40

two frames are connected by pivoted rods so that they remain in position parallel to one another during the experiment.

When the experimental wire is loaded, the end of the spirit level supported on the micrometer screw is lowered in position, and the bubble of the spirit level moves to the other side. *The screw is then worked so as to move upwards until the bubble moves back to the central position.* The readings of the micrometer screw give the length by which the screw has been moved upwards, and consequently also the length by which the experimental wire is extended due to loading.

(ii) An optical lever may also be used instead of the spirit level and micrometer screw for measuring extension. The optical lever consists of a small rectangular strip of metal S which is supported on two legs L and M on one side and one leg N on the other, the pointed ends of the legs forming an isosceles triangle. A small plane mirror is mounted vertically on the strip, parallel to the plane of the two legs L and M at one end. The two legs rest on the frame hanging from the comparison wire and the leg N rests on the frame hanging from the experimental wire. When this frame is lowered due to loading in the experiment, the point of the leg N is lowered in position (to N′ say) and consequently the optical lever and the mirror on it are tilted. If the angle of the tilt be θ, and if the distance of N from the midpoint K of LM be d, (Fig. 43), the extension, $e = d.\theta$.

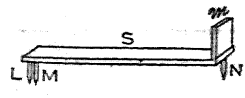


Fig. 41

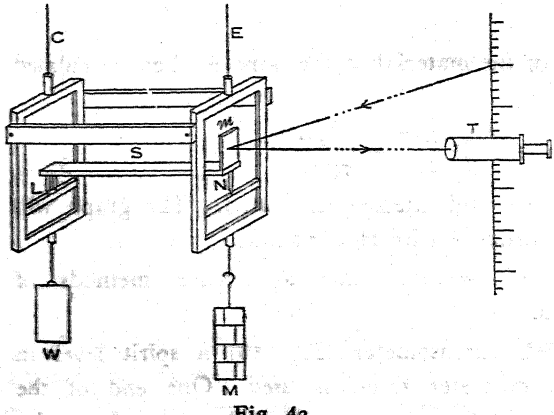


Fig. 42

For measuring θ, a scale and a reading telescope are used. A millimetre scale is supported vertically in front of the mirror at a distance, D, of about a metre and its image in the plane mirror is seen through the reading telescope, T, which is fixed by the side of the scale. The readings which coincide with the horizontal cross-wire before and after loading are observed. If the difference between the two readings is x, and the distance between the mirror and the scale is D, the angle through which the reflected beam turns is equal to 2θ and

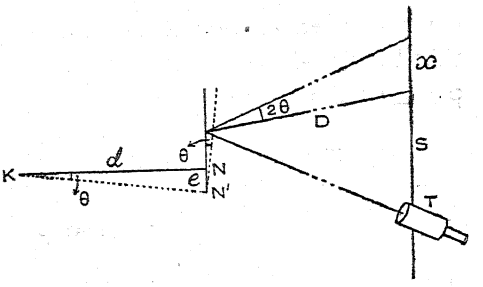


Fig. 43

$$2\theta \times D = x \quad \text{or} \quad \theta = \frac{x}{2D} \text{ (Fig. 43)}$$

$$\therefore \text{ Extension } = e = d.\theta = \frac{d\,x}{2D}.$$

TORSION

74. A cylindrical wire (or rod) is said to be under **torsion** when it is fixed at one end and twisted about its axis by means of a couple applied at the other end. If such a wire is considered to be divided into a number of thin coaxial cylindrical layers, each one of them undergoes a shear as can be easily seen below.

Let a cylindrical wire having a length l and radius of cross-section 'a' be fixed at its upper end and twisted about its axis by means of a couple applied at its lower end.

Consider a coaxial cylindrical layer of radius r and thickness dr within the cylinder. Let a vertical line CD (Fig. 44) be drawn parallel to its axis before twisting. If now the cylindrical layer is cut open along CD and spread out, it will be a rectangle, CDML (Fig. 45). But when the twist is applied, the line CD occupies the position CE. If the cylindrical layer is

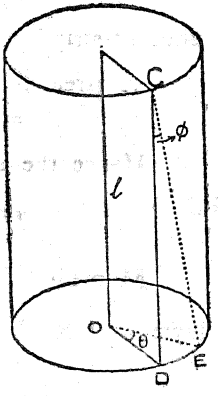


Fig. 44

now cut open along CE and spread out, it will be a parallelogram CENL. Obviously, this is a case of shear and the shear strain $\phi = \angle DCE = DE / l$.

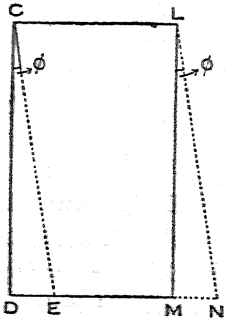


Fig. 45

But $DE = r\theta$. (Fig. 46)

(θ being the angle of twist on the lower end)

$$\therefore \text{Shear } \phi = \frac{DE}{CD} = \frac{r\theta}{l} \quad \ldots \quad (1)$$

When the wire is twisted, *each circular cross-section of the wire is rotated about its axis by an amount proportional to its distance from the fixed end and thus,* the twist per unit distance from the fixed end $= \frac{\theta}{l}$.

If n is the modulus of rigidity of the material of the wire,

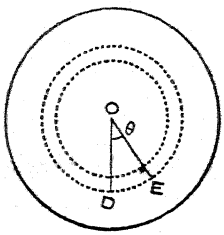


Fig. 46

$$n = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{\text{shearing stress, S}}{\phi}.$$

$$\therefore S = n \cdot \phi = n \cdot r\frac{\theta}{l} \text{ from (1)} \quad \ldots \quad (2)$$

But the shearing stress, S

$$= \frac{\text{shearing force, } f}{\text{area on which the force acts}}.$$

$\therefore$ Shearing force, f = S $\times$ area on which f acts.

The area over which the shearing force f acts is the area of the annular strip between the circles of radii r and $r + dr$.

$\therefore$ Area of the strip $= \pi (r + dr)^2 - \pi r^2$
$= 2\pi r \cdot dr$ approximately.

Hence the shearing force f = S $\times 2\pi r \cdot dr$.

$$= n.r\frac{\theta}{l} \times 2\pi r\, dr \quad \ldots \quad \ldots \quad (3)$$

Moment of the force, f about the axis of the cylindrical layer $= f \times OE = \frac{nr\theta}{l} \times 2\pi r\, dr \times r = \frac{2\pi n}{l}\theta\, r^3 dr \quad \ldots \quad (4)$

The modulus of rigidity of the material of the wire can thus be obtained from the above relation.

(b) **Static Torsion apparatus : Horizontal pattern**

The horizontal pattern described below is commonly used in the laboratories for the measurement of the rigidity modulus (n) of any material supplied in the form of a rod.

The experimental rod (usually of length 50 cm. and of diameter 0·5 cm.) is secured firmly at one end A and fitted into the axle of a wheel W at the other end B. The wheel is provided with a grooved edge over which passes a tape, in clockwise or in anti-clockwise direction, carrying a weight hanger at its free end. Two vertical pointers, P_1 and P_2 clamped at two points Q and R on the rod, move over two circular scales S_1 and S_2 respectively. The scales are marked in degrees with centre zero.

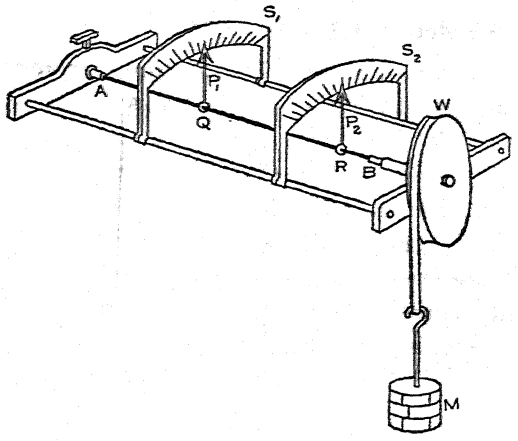


Fig. 48

The tape is wound over the wheel, say, **in clockwise direction** and the weight hanger of known mass suspended from its free end. The pointers P_1 and P_2 are clamped at convenient distances, l_1 cm. and l_2 cm. from the fixed end A of the rod. The pointers are initially adjusted to read zero on the scales respectively, when no weights are put on the hanger. The load on the hanger is increased in steps of 200 gm. at a time to a maximum of 2000 gm. The positions of the pointers P_1 and P_2 on the scales are noted after each addition of the load. The load is then reduced in the same steps of 200 gm. and the corresponding readings on the two scales respectively are noted again for every load.

The tape is next wound over the wheel **in anti-clockwise direction** so as to apply the twisting couple in the opposite direction. The experiment is repeated and the readings on the scales, when the load is increased and also when the load is decreased, are noted as before.

The average of the four readings obtained for each load and for each scale is calculated. Then the mean twists a_1 and a_2 (in degrees) undergone by the rod at the points Q and R for a load of 100 gm. are also calculated.

The observations are tabulated thus:

Load	Reading on S_1				Mean	Twist	Twist a_1 for 1000 gm.
	Clockwise		Anti-clockwise				
	Load increasing	Load decreasing	Load increasing	Load decreasing			

The observations from the second scale are also similarly tabulated and the mean value a_2 for the same load calculated.

If a load M gm. is suspended from the free end of the tape, the twisting couple $= Mg.r$ .... (1)

(r being the radius of the wheel).

If this couple twists the rod through an angle θ (in radians), at a point distant l from the fixed end, the restoring couple developed inside the rod $= \frac{\pi n a^4}{2l}\theta$ .... (2)

The restoring couple balances the twisting couple so that $Mgr = \frac{\pi n a^4}{2l}\theta$ or $n = \frac{2Mg \cdot rl}{\pi a^4 \theta}$ .... (3)

If now a_1 and a_2 are the twists produced in degrees at two points on the rod which are l_1 and l_2 cm. respectively from the fixed end A of the rod, then

$$n = \frac{2Mgrl_1}{\pi a^4} \times \frac{180}{\pi a_1} \quad \ldots \quad \ldots \quad (4)$$

$$\left(\because \alpha_1 \text{ in degrees} = \frac{\pi\alpha_1}{180} \text{ in radians} \right)$$

$$\text{and } s = \frac{2Mgrl_2}{\pi a^4} \times \frac{180}{\pi\alpha_2} \quad \ldots \quad \ldots \quad (5)$$

$$\left(\because \alpha_2 \text{ in degrees} = \frac{\pi\alpha_2}{180} \text{ in radians} \right)$$

$$\text{or } n = 360\,\frac{Mgr}{\pi^2 a^4}\left(\frac{l_2 - l_1}{\alpha_2 - \alpha_1}\right) \quad \ldots \quad \ldots \quad (6)$$

Thus measuring the distances l_1 and l_2 (of the pointers P_1 and P_2 from the fixed end of the rod A), $2a$ (the diameter of the rod) and $2r$ (the diameter of the wheel), the modulus of rigidity, n, of the material of the rod can be calculated.

The value $2a$ should be accurately measured since 'a' occurs in the fourth power in the relation used.

It may be noted that the *errors due to the slipping of the ends A or B*, if any, is elimnated by taking the ratio $\frac{l_2 - l_1}{\alpha_2 - \alpha_1}$ instead of $\frac{l_2}{\alpha_2}$ or $\frac{l_1}{\alpha_1}$ in the calculations.

Similarly, *the error due to the eccentricity of the wheel*, if any, is elimlnated by applying the couple in both clockwise and anti-clockwise directions and taking the average of the readings for each load.

76. Dynamical method of Determining 'n': Torsion pendulum.

The torsion pendulum consists of a heavy body suspended from a fixed point by means of a wire. When the body is turned through an angle about the wire as the axis, the wire is twisted and the torsional couple, brought into action, sets the body in oscillation in the horizontal plane. The body is then said to execute **torsional oscillations.**

When, in its oscillation, the twist on the lower end of the wire is θ, the restoring couple $= \frac{\pi n a^4}{2l}\theta = c.\,\theta$.

(c being the couple per unit twist.)

If I is the moment of inertia of the body, the angular acceleration

of the system $= \dfrac{\text{couple acting on the system}}{\text{moment of inertia}} = \dfrac{c\theta}{I}$ (from Art. 19).

It is seen further that the acceleration is directed towards the position of equilibrium.

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{c}{I}\theta$$

$$\text{or} \quad \frac{d^2\theta}{dt^2} + \frac{c}{I}\theta = 0$$

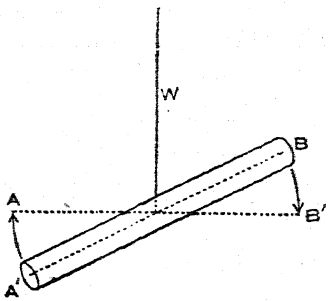


Fig. 49

∴ The body has Simple Harmonic Motion and the period, T is given by relation $T = 2\pi\sqrt{\dfrac{I}{c}}$.

But $c = \dfrac{\pi n a^4}{2l}$.

$$\therefore T = 2\pi\sqrt{\frac{I}{\pi n a^4 / 2l}}.$$

$$\text{Hence } T^2 = \frac{8\pi^2 I l}{\pi n a^4} \quad \text{or} \quad n = \frac{8\pi I l}{T^2 a^4}$$

The modulus of rigidity, n, of a wire can be determined by suspending a body of known moment of inertia from it and making it perform torsional oscillations. The period of oscillation T is determined and the length (l) and the radius of cross-section, a, of the wire are measured. *Care is exercised in measuring 'a' since it occurs in the fourth power.*

I, the moment of inertia of the body, is calculated from its mass and its dimensions.

[If the body is in the form of a cylindrical bar, its moment of inertia $I = M\left(\dfrac{l^2}{12} + \dfrac{r^2}{4}\right)$, l being its length and r the radius of cross-section; and if it is in the form of a disc or a cylinder of radius r having its axis parallel to the axis of rotation, $I = M\dfrac{r^2}{2}$].

Substituting the different values in the expression, $n = \dfrac{8\pi I l}{T^2 a^4}$ the modulus of rigidity, n, is calculated.

77. For greater accuracy, the body used in this experiment may be in the form of a cylindrical rod on which can be fixed two cylindrical weights at any desired positions. The experiment is conducted fixing the weights at equal distance d_1 on either side of the centre of the rod. The period T_1 is determined.

If I_0 is the moment of inertia of the rod alone,
m the mass of each of the weights and

i_0 the moment of inertia of each weight about a parallel axis passing through its centre of gravity, the moment of inertia of the whole system,

$$I_1 = I_0 + i_0 + md_1^2 + i_0 + md_1^2 = I_0 + 2i_0 + 2md_1^2 \qquad (1)$$

If T_1 is the period of oscillation we have

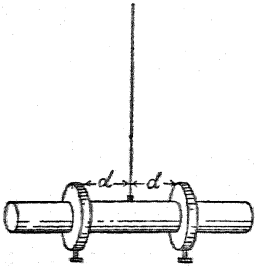


Fig. 50

$$T_1^2 = \frac{8\pi l}{a^4 n} \cdot I_1 \qquad \ldots \qquad (2)$$

The weights are now shifted to a distance d_2 from the centre of the rod and the corresponding period T_2 of the torsional oscillation is again determined.

Now, $I_2 = I_0 + 2i_0 + 2md_2^2$ .... (3)

and $T_2^2 = \frac{8\pi l}{a^4 n} \cdot I_2$ ... (4)

From (2) and (4) we get

$$T_2^2 - T_1^2 = \frac{8\pi l}{a^4 n}(I_2 - I_1) = \frac{8\pi l}{a^4 n} 2m(d_2^2 - d_1^2)$$

$$\text{or} \quad n = \frac{16\pi lm(d_2^2 - d_1^2)}{a^4(T_2^2 - T_1^2)}.$$

Using this relation, the value of n can be determined accurately.

78. The value for 'n' measured from statical experiments will be slightly lower than that obtained from dynamical experiments. This is due to the fact that the twist in statical experiments increases a little with time while in the dynamical experiments the time allowed is short and no yielding of this type is produced.

79. Moment of inertia of a body by torsional oscillations

The period T_1 of the torsional pendulum with a body of known moment of inertia I_0 is determined.

$$\text{Now } T_1^2 = \frac{8\pi I_0 l}{na^4} \quad \ldots \quad \ldots \quad (1)$$

The given body is then attached to the body used above and the period of oscillation T_2 is again determined.

If I is the moment of inertia of the given body, we have

$$T_2^2 = \frac{8\pi l}{na^4}(I_0 + I) \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2) we get

$$\frac{T_2^2}{T_1^2} = \frac{I_0 + I}{I_0} = I + \frac{I}{I_0}$$

$$\therefore \frac{I}{I_0} = \frac{T_2^2 - T_1^2}{T_1^2} \quad \text{or} \quad I = \frac{T_2^2 - T_1^2}{T_1^2} \cdot I_0.$$

The moment of inertia of the given body is thus calculated using this relation.

80. Torsion Balance

It may be sometimes convenient to measure a force by making it twist a wire and determining the angle of twist produced. An arrangement in which a force is measured in this manner is called a **torsion balance.** This method was employed by Cavendish for determining the force of Gravitational attraction between two bodies as will be explained under **'Gravitation'.**

It was also employed by Coulomb to verify inverse square law in Electro-statics.

BENDING

81. Bending of Beams

A beam is a body of uniform cross-section, the length of which is very large compared to its thickness. When such a beam is fixed at one end A and loaded at the other end B, it is called a **cantilever.**

Let a cantilever, AB of negligible weight, carry a weight W at the free end B. Since the beam is in equilibrium a force of reaction, R (= W) will develop at A and it acts upwards. These two forces W and R, being equal in magnitude and opposite in direction, form a couple. This couple is responsible for the

bending of the beam. The *plane of bending is the plane in which the bending takes place* and the bending couple acts in this plane. In the case, illustrated in Fig. 51, the plane of the paper is the plane of bending.

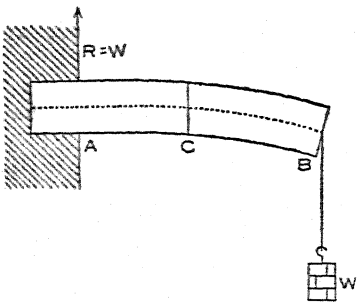


Fig. 51

Now, the force R acts at A, parallel to the cross-sections of the beam and it tends to shear the beam across the cross-section. Not only does a shearing force act at A in this manner but shearing forces act also parallel to every cross-section of the beam. For, if the equilibrium of any part say BC of the beam is considered, there should necessarily be a force R = W acting vertically upwards at C in order to balance the force W acting at B (Fig. 52).

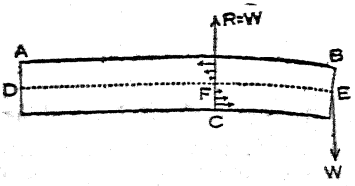


Fig. 52

Taking the equilibrium of BC into consideration, it is seen that the forces W at B and R at C will not by themselves be able to maintain the equilibrium of BC since they form a couple, the moment of which is equal to W × BC.

For equilibrium, the external bending moment (= W × BC) must be balanced by an equal and opposite bending moment. This balancing bending moment is provided by the elastic reactions brought into play within the material of the beam and is called the **internal bending moment.**

When a beam is bent as in Fig. 52, longitudinal filaments in the upper part of the beam are lengthened and those in the lower part are shortened. There must also, therefore, be some filaments somewhere between the upper and lower parts of the beam which are neither elongated nor compressed but retain their initial length. All such filaments lie in a surface called the **neutral surface.** When the beam is not strained, the neutral surface is a plane surface. The plane of bending is perpendicular to the neutral surface and the longitudinal axis of the neutral surface is called the **neutral axis.** The neutral axis

passes through the centres of all the cross-sections of the beam. Thus, if DE is the neutral axis of a bent beam AB, the longitudinal filament along the neutral axis, DE, retains its initial length; filaments above DE are elongated while those below DE are compressed. It is obvious from the figure that *the length of any filament increases or decreases in proportion to its distance away from the neutral axis, DE.*

Coming back to the part BC of the beam, the filaments in BC above FE are elongated and should therefore be subjected to pulling forces as shown in Fig. 52 which tend to elongate them. The filaments below FE are shortened and should therefore be subjected to pushing forces tending to compress them. If we assume that the pulling and pushing forces are proportional to the distances at which they act from the neutral surface, it follows that the pushing and pulling forces form pairs of forces having equal values and each pair forms a couple. It is also seen that the moments of these couples are opposite to the external bending moment. *The resultant of the moments of all these internal couples is called the* **internal bending moment** and it is equal to the external bending moment since the beam is in equilibrium.

82. Internal bending moment

Let KM and LN be the traces of two cross-sections of a bent beam which are near each other and perpendicular to the neutral surface PQ. Filaments above PQ are elongated while those below PQ are compressed but the filament PQ remains unchanged in length. The normal length of all the filaments before strain is therefore given by PQ.

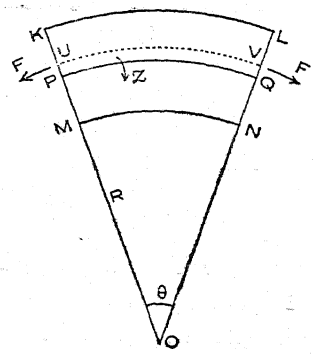


Fig. 53

Produce KM and LN to meet at O, the centre of curvature of PQ.

Let R be the radius of curvature of PQ and θ the angle subtended by it at O so that

$$PQ = R\theta \quad \ldots \quad \ldots \quad (1)$$

Now, consider any other filament UV close to PQ and at a distance Z from it.

Then $UV = OU \times \theta = (OP + PU)\theta$

$= (R + Z)\theta$ ... ... (2)

Since the normal length of all the filaments is the same and equal to PQ, the extension suffered by the filament UV is equal to $(UV - PQ)$.

$\therefore$ Longitudinal strain suffered by $UV = \dfrac{UV - PQ}{PQ}$

$$= \frac{(R+Z)\theta - R\theta}{R\theta} = \frac{Z}{R}.$$

If F is the magnitude of the tensile force acting on the filament, UV, and a, the area of cross-section of the filament,

the tensile stress $= \dfrac{F}{a}$ and $\dfrac{F/a}{Z/R} = Y.$

(where Y is the Young's modulus of the material of the bar.)

$\therefore F = \dfrac{Y}{R}.Z.a.$

The moment of F about a line passing through P and perpendicular to the plane of the paper, i. e., about a line GH perpendicular to the plane of bending and passing through the centre of the cross-section of the beam

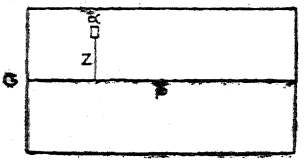


Fig. 54

$= F \times Z$

$= \dfrac{Y}{R}.a.Z^2.$

The internal bending moment is equal to the sum of the moments of all such forces as F (above and below the neutral surface) acting on different filaments.

$\therefore$ Internal bending moment $= \Sigma \dfrac{Y}{R}.a\,Z^2 = \dfrac{Y}{R}\Sigma\, a\,Z^2.$

The quantity ΣaZ^2 is called the **geometrical moment of inertia** of the cross-section of the bar about a line GH (passing through the centre of the cross-section of the bar and perpendicular to the plane of bending) and written as equal to AK^2,

(A being the area of cross-section of the bar and K the radius of gyration of the beam about the said axis)

$\therefore$ Internal bending moment $= \dfrac{YAK^2}{R} = \dfrac{YI}{R}.$

(I being the geometrical moment of inertia $= AK^2$)

Thus when any beam AB is fixed at A and loaded with a weight W at B, any part of it, BC (Fig. 52) is in equilibrium under the external bending moment ($= W \times BC$) and the internal bending moment $\left(\frac{YI}{R} \right)$.

$$\therefore W \times BC = \frac{YI}{R}.$$

The external bending moment required to produce unit radius of curvature is called the **flexural rigidity** of the bar and it is equal to YI.

[Geometrical moment of inertia for a bar of

(1) rectangular cross-section $= bd^3 / 12$

b being the breadth and d the thickness of the bar and

(2) circular cross-section $= \frac{\pi r^4}{4}$

r being the radius of cross-section of the bar]

83. General expression for the radius of curvature, R

Let OMN be a curve having a large radius of curvature and let M and N be two points on it which are very close together.

Draw OQ, MS and NT tangents at O, M and N respectively.

Draw MC and NC, perpendicular to the tangents at M and N respectively to meet at C. Then MC = NC = R, where R is the radius of curvature of MN.

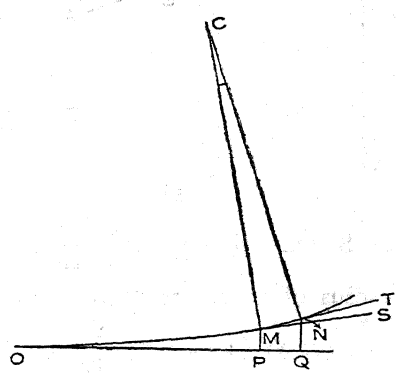


Fig. 55

Since R is large,

$$R \times \angle MCN = MN \quad \ldots \quad (1)$$

But $\angle MCN$ = Slope of the tangent at M − Slope of the tangent at N

$$= \left(\frac{dy}{dx} \text{ at M} - \frac{dy}{dx} \text{ at N} \right).$$

$$\therefore R \times \left(\frac{dy}{dx} \text{ at } M - \frac{dy}{dx} \text{ at } N \right) = MN \quad \ldots\ldots \quad (2)$$

If MP and NQ are drawn perpendiculars to OQ, then MN = PQ approximately.

$$\therefore PQ = R \left(\frac{dy}{dx} \text{ at } M - \frac{dy}{dx} \text{ at } N \right)$$

$$\text{and } \frac{1}{R} = \left(\frac{dy}{dx} \text{ at } M - \frac{dy}{dx} \text{ at } N \right) / PQ.$$

If $PQ = \delta x$

$$\frac{1}{R} = \left(\frac{dy}{dx} \text{ at } M - \frac{dy}{dx} \text{ at } N \right) / \delta x$$

$$= \frac{d^2y}{dx^2} \text{ at } M \left(\text{since } \frac{dy}{dx} \text{ is small} \right).$$

84. Depression of the loaded end of a cantilever

Let OA be a cantilever of length l and of negligible mass fixed at O and loaded with a weight W ($= mg$) at A. Let OX and OY be chosen as axes along and perpendicular to OA′, the unstrained position of the beam.

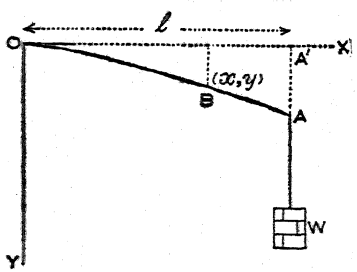


Fig. 56

Consider any point B on the neutral axis having the co-ordinates (x, y). The load W at A gives rise to a force of reaction W at B so that the external bending moment =

$$W \times AB = W(l - x) \quad \ldots \quad (1)$$

This external moment is balanced by an internal bending moment at B which is equal to $\frac{YI}{R}$ as shown in Art. 82, (where R is the radius of curvature at B.)

$$\therefore W(l - x) = \frac{YI}{R} \quad \ldots\ldots \quad \ldots\ldots \quad (2)$$

But $\frac{1}{R} = \frac{d^2y}{dx^2}$ so that

$$YI \cdot \frac{d^2y}{dx^2} = W(l - x) \quad \ldots \quad \ldots\ldots \quad (1a)$$

By integrating this equation with respect to x, we get

$$\text{YI.}\ \frac{dy}{dx} = W\left(lx - \frac{x^2}{2}\right) + C_1 \quad \ldots \quad \ldots \quad (2)$$

(C_1 being the constant of integration).

To find the value of C_1, we note that at O (i.e., at $x=0$) $\frac{dy}{dx}=0$. Substituting these values in equation (2) we get $C_1=0$.

$$\therefore\ \text{YI.}\ \frac{dy}{dx} = W\left(lx - \frac{x^2}{2}\right) \quad \ldots \quad \ldots \quad (3)$$

Integrating equation (3) with respect to x, once again, we obtain

$$\text{YI.}\ y = W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_2 \quad \ldots \quad \ldots \quad (4)$$

(C_2 being the constant of integration now.)

In order to find C_2, we again note that at O, (i.e., at $x=0$) $y=0$.

Substituting in (4) we get $C_2 = 0$.

$$\therefore\ \text{YI.}\ y = W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$$

$$\text{or}\quad y = \frac{W}{YI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \quad \ldots \quad \ldots \quad (5)$$

Thus the depression Y at any point B, distant x from 'O', is given by the equation, (5). The depression of the free end of the beam, A may now be found.

If $y=e$ at A, we have $x=l$ and

$$= \frac{W}{YI}\left(\frac{l \cdot l^2}{2} - \frac{l^3}{6}\right) = \frac{W \cdot l^3}{3YI} \quad \ldots \quad \ldots \quad (6)$$

$$\therefore\ Y = \frac{Wl^3}{3\,Ie}.$$

85. Depression at the mid-point of a beam loaded at the middle

Let a beam AB be supported on two knife-edges C and D near its ends and loaded at a point E, which is the mid-point of CD.

The load Mg (say) at E gives rise to forces of reaction,

$$\frac{Mg}{2} \text{ at C and } \frac{Mg}{2} \text{ at D.}$$

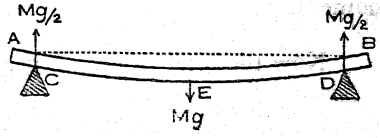


Fig. 57

It may be noticed that the part of the beam at E is horizontal so that each half of the beam may be considered as a cantilever clamped at E and displaced upwards by a force, $\frac{Mg}{2}$ at C (or D).

The relative elevation 'e' of the points C or D over E may be obtained by substituting in equation (6) above.

$W = \frac{Mg}{2}$ and $l = \frac{L}{2}$, L being the length of the beam between the knife-edges.

$\therefore$ Depression of E = Relative elevation, e, of C or D over E

$$= \frac{Mg/2.\ (L/2)^3}{3YI} = \frac{MgL^3}{48YI}$$

$$\therefore e = \frac{MgL^3}{48\,YI} \text{ and } Y = \frac{MgL^3}{48\,Ie}.$$

86. Uniform bending of a beam : Statical Method

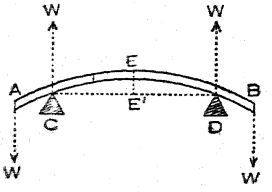


Fig. 58

Let a beam AB be supported symmetrically on two knife-edges C and D, at the same level. It is loaded symmetrically with equal weights, (W, W) beyond the knife-edges, at A and B, so that AC = BD.

The reactions on the knife-edges will then be W and W acting vertically upwards at C and D.

Considering a point E on the beam, the forces W at A and W at C exert moments at E which are respectively equal to (W × AE) and (−W·CE). The net moment of these forces which is the external bending moment on the part AE of the bar.

$$= W \cdot AE - W \cdot CE = W \cdot AC.$$

$$= W \times p \text{ (where AC} = p) \quad \ldots\ldots (1)$$

This must be balanced by the resultant moments of the forces which act across the cross-section at E, i.e., the internal bending moment.

Now the internal bending moment $= \frac{YI}{R}$ .... (2)

$$\therefore W.p = \frac{YI}{R} \quad \ldots\ldots (3)$$

Since for a given load, W, Y, p and I are constant, R must also be constant; i.e, the beam is bent uniformly so that CD forms an arc of a circle of radius R.

If 'e' is the elevation of the mid-point of CD above its normal position, and l is the length of the beam between the knife-edges C and D,

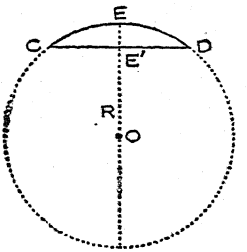


Fig. 59

we have, $e\,(2R - e) = \left(\frac{l}{2}\right)^2$

or $2Re = \frac{l^2}{4}$ (neglecting e^2 since e is very small compared to l)

i.e., $\frac{1}{R} = \frac{8e}{l^2}$.

Substituting for $\frac{1}{R}$ in equation (3), we have,

$$W.p = \frac{YI.\,8e}{l^2} \quad \text{or} \quad e = \frac{Wpl^2}{8YI} \text{ and } Y = \frac{Wpl^2}{8Ie}.$$

87. Oscillations of a loaded bar

Let OA be a cantilever of negligible mass fixed at O and loaded by a weight W at A. It has already been shown in Art. 84 that, in such a cantilever, the loaded end is lowered by an amount, e given by the relation,

$$Y = \frac{Wl^3}{3\,Ie}$$

where W is the force acting vertically downwards.

$$\therefore W = \frac{3YI}{l^3} \cdot e \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

If the beam is now set in oscillations, the loaded end A moves up and down about A. If, at any instant in its oscillation, the depression of the loaded end A is y from the normal position, the force F acting on the beam is given by the relation,

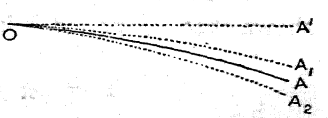


Fig. 60

$$F = \frac{3YI}{l^3} \cdot y \quad \ldots \quad \ldots \quad (2)$$

Work needed for a further depression, dy

$$= F \cdot dy = \frac{3YI}{l^3} \cdot y \cdot dy. \quad \ldots \quad (3)$$

∴ Work done in depressing the end A by y from its rest position $= \int_0^y \frac{3YI}{l^3} y \cdot dy = \frac{3YI}{l^3} \cdot \frac{y^2}{2}$.

The work so spent is stored up in the beam as potential energy.

∴ The potential energy of the beam when the depression of the loaded end is 'y' $= \frac{3YI}{l^3} \cdot \frac{y_2}{2}$ ... (4)

If m is the total mass suspended at A, the kinetic energy of the beam in its oscillation $= \frac{1}{2} m \left(\frac{dy}{dt} \right)^2$ ... (5)

(mass of the beam being negligible)

Applying the principle of conservation of energy,

we have, $\frac{1}{2} \cdot m \left(\frac{dy}{dt} \right)^2 + \frac{1}{2} \frac{3YI}{l^3} \cdot y^2 = \text{constant}$ ... (6)

On differentiating equation (6) with respect to time,

we get, $\frac{1}{2} m \frac{2dy}{dt} \cdot \frac{d^2y}{dt^2} + \frac{1}{2} \cdot \frac{3YI}{l^3} 2y \frac{dy}{dt} = 0.$

$$\therefore m \cdot \frac{d^2y}{dt^2} + \frac{3YI}{l^3} \cdot y = 0.$$

This equation shows that the oscillations of the beam are simple harmonic and its period of oscillation T is given by the relation,

$$T = 2\pi \sqrt{\frac{m}{3YI/l^3}} = 2\pi \sqrt{\frac{ml^3}{3YI}}.$$

If, however, the mass of the beam is not negligible, it can be shown that the period T is given by the relation,

$$T = 2\pi \sqrt{\frac{l^3}{3YI} \left(m + \frac{33}{140} M \right)}.$$

(M, being the mass of the beam.)

DETERMINATION OF 'Y'

In experiments on the determination of Y by the bending of beams, *it is convenient to use beams of rectangular cross-section, although beams of circular cross-section may also be used.*

88. Non-uniform bending : Double Cantilever

A beam AB of rectangular cross-section is supported on two knife-edges C and D in a horizontal level (Fig. 57). A weight hanger (of weight, w gm.) is suspended by means of a loop of thread from the point E on the beam exactly midway between the knife-edges. To obtain the depression of E on loading, a pin is fixed vertically by some wax at E and a travelling microscope, provided with vertical traverse, is focussed on the pin. The microscope is adjusted so that the horizontal cross-wire coincides with the tip of the pin and the reading on the vertical scale of the travelling microscope is noted.

The load on the beam is then increased by adding weights on the hanger in equal steps of 500 gm up to a maximum of 3,500 gm. The microscope is adjusted each time to make the horizontal cross-wire coincide with the tip of the pin and the reading on the vertical scale is noted in each case. Similarly, readings are obtained when the load is decreased also in equal steps of 500 gm.

The results are tabulated as follows :

S. No.	Load in gm.	Readings on the scale			Depression	Depression for 2000 gm.
		Load increasing	Load decreasing	Mean		
1	w			a		
2	$(w+500)$			b	$b-a$	
3	$(w+1000)$			c	$c-a$	$f-a$
4	$(w+1500)$			d	$d-a$	$g-b$
5	$(w+2000)$			f	$f-a$	$h-c$
6	$(w+2500)$			g	$g-a$	$i-d$
7	$(w+3000)$			h	$h-a$	
8	$(w+3500)$			i	$i-a$	

Mean, e =

The mean depression 'e' of the mid-point E (Fig. 57) for a load of M gm. (say 2,000 gm.) is calculated by taking the average of the values in the last column.

The breadth of the beam b is measured with a slide calipers. The thickness is measured at different points of the beam with a screw gauge and the average thickness, d, obtained. The geometrical moment of inertia I is then calculated from the relation,

$$I = \frac{bd^3}{12}.$$

The length of the beam (L) between the knife-edges is also measured. Young's modulus of the material of the bar is then calculated from the relation $Y = \frac{Wl^3}{48Ie} = \frac{Mgl^3}{4bd^3e}$

A graph may also be plotted between the load (on the y-axis) and depression (on the x-axis). It will be a straight line. The depression, e, for any load M may be read off from the graph and Y calculated by substituting the value in the above expression.

89. Single Cantilever

'Y' of the material of the beam can also be found by using it as a single cantilever. The beam, for this purpose, is fixed at one end A and loaded at the other end B. The experiment may be conducted as described above and Y calculated from the relation $Y = \frac{4Mgl^3}{bd^3e}$.

90. Gradient method

From equation 3 in Art. 84 we have,

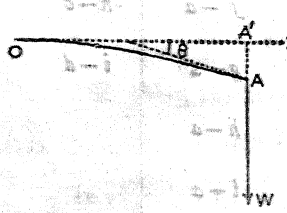


Fig. 61

$$YI.\frac{dy}{dx} = W\left(lx - \frac{x^2}{2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{W}{YI}\left(lx - \frac{x^2}{2}\right).$$

When $x = l$, (i.e., at the free end of the bar)

$$\frac{dy}{dx} = \frac{Wl^2}{2YI} \quad \ldots \quad (1)$$

Now since $\frac{dy}{dx} = \tan\theta$ (where θ is the angle which the tangent, drawn to the bent bar at its free end, makes with the horizontal) and since θ is small

$$\frac{dy}{dx} = \tan\theta = \theta = \frac{Wl^2}{2YI}. \quad \ldots \quad \ldots \quad (2)$$

To determine θ, a mirror strip is attached to the bar above the point where the load is suspended and the angle through which the mirror turns ($= \theta$) is determined using a scale and a reading telescope. A millimetre scale is supported vertically at a distance (D) of about a metre and its image in the plane mirror is seen through the reading telescope which is fixed by the side of the scale.

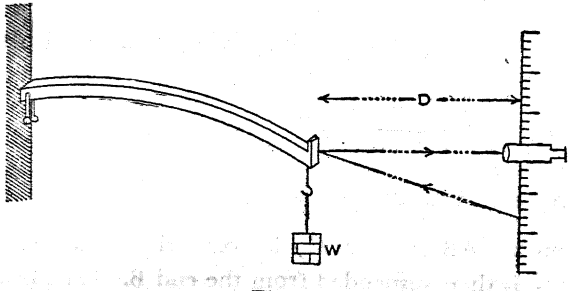


Fig. 62

The reading on the scale which coincides with the horizontal cross-wire is taken before the bar is loaded. The bar is then loaded and the reading is again taken.

If x is the difference between the two readings

$$\theta = \frac{x}{2D} \quad \ldots\ldots \text{(vide Art. 73).}$$

Thus using the relation (2), the Young's modulus of the material of the beam is calculated.

91. Uniform bending

The given beam of rectangular cross-section AB is supported *symmetrically* on two knife-edges C and D (Fig. 58). Two weight hangers of equal weight are hung from the ends of the beam (or from points on the bar at equal distances from the ends). The load on the hangers is increased or decreased equally. A vertical pin is arranged

at the mid-point of the beam and a travelling microscope focussed on the tip of the pin.

The experiment is conducted as in the above experiment (1) by increasing the load on the two hangers equally in steps of 500 gm. up to a maximum of 3,500 gm. and then decreasing them equally in the same steps. The microscope in each case is focussed against the tip of the pin and the reading on the vertical scale is taken. The observations are tabulated as in experiment (1) and the mean value of the elevation (e) for a load of m gm. (say 2,000 gm.) is obtained.

The length of the beam l between the knife-edges, p, the distance of the point where the weight is suspended from the near knife-edge, b the breadth and d the thickness of the beam are measured as before.

Young's modulus of the material of the beam is calculated from the relation

$$Y = \frac{Wl^2p}{8eI} = \frac{3}{2}\frac{mgl^2p}{bd^3e}.$$

92. Oscillations of a Cantilever

The given bar AB of mass M, is fixed rigidly at A. A certain load of m_1 gm. is then suspended from the end B. The beam is set in transverse oscillations and the time for 25 oscillations is obtained twice. The mean period of oscillation T_1 is calculated from the mean of the two observations.

Young's modulus of the material of the beam, Y is then calculated from the relation

$$T_1 = 2\pi\sqrt{\left(m_1 + \frac{33M}{140}\right) l^3 / 3YI}$$

$$\text{or, } Y = \frac{4\pi^2 l^3}{3I}\left(\frac{m_1 + \frac{33M}{140}}{T_1^2}\right) \quad \ldots \quad (1)$$

To eliminate 'M' from the expression, the period of oscillation of the beam T_2 is obtained when it is loaded with a weight of m_2 gm.

$$\text{Now } Y = \frac{4\pi^2 l^3}{3I}\left(\frac{m_2 + \frac{33M}{140}}{T_2^2}\right) \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2),

$$Y = \frac{4\pi^2 l^3}{3I} \cdot \frac{m_2 - m_1}{T_2^2 - T_1^2} = \frac{16\pi^2 l^3}{bd^3} \cdot \frac{m_2 - m_1}{T_2^2 - T_1^2}.$$

'Y' is calculated using this expression.

93. Determination of Y, n and σ with Searle's apparatus

(i) Determination of Y: Uniform bending—Dynamic method.

Searle's apparatus consists of two exactly similar brass rods AB and CD. A wire of length l (about 30 cm.) is rigidly fixed to the rods by means of screws at their centres E and F respectively. The

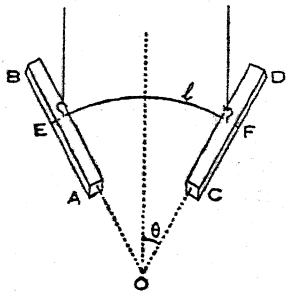


Fig. 63

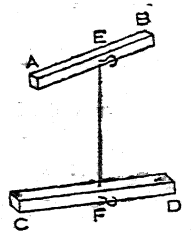


Fig. 64

rods are then suspended by two parallel strings so that they lie in the same horizontal plane and the wire is straight when the rods are parallel to each other.

The ends A and C of the rods are drawn together a little, thus forcing the wire to bend into an arc of a circle. If now the rods are released, they begin to oscillate on account of the couple exerted on them by the bent wire. Since the bending moment is uniform throughout, the wire will be in the form of an arc of a circle at any instant.

If at any instant in its oscillation, the rod AB or CD makes an angle θ with its position of rest and the wire be at the moment bent into an arc of a circle of radius, R, the bending moment due to the wire $= \frac{YI}{R}$ (where Y is the Young's modulus of the material of the wire and I, the geometrical moment of inertia of the cross-section of the wire).

Now, $R \cdot 2\theta = l$ or $\frac{I}{R} = 2\frac{\theta}{l}$

and I, the geometrical moment of inertia of a wire $= \frac{\pi a^4}{4}$, i.e., $I = \frac{\pi a^4}{4}$ ('a' being the radius of cross-section of the wire.)

$\therefore$ The bending moment $= \frac{Y\pi a^4}{2l} \cdot \theta = c.\ \theta.$

(c being the couple per unit angular displacement.)

This couple, which is proportional to θ tends to restore either rod to its equilibrium position. Hence the motion is simple harmonic and the period of oscillation, T (of either rod AB or CD)

$$= 2\pi\sqrt{\frac{I}{c}}$$

(where I is the moment of inertia of either rod.)

$$\therefore T = 2\pi\sqrt{\frac{I}{c}} = 2\pi\sqrt{\frac{I \cdot 2l}{\pi Y a^4}} \quad \text{or} \quad T^2 = \frac{8\pi I l}{Y a^4}$$

$$\therefore Y = \frac{8\pi I l}{T^2 a^4} \quad \ldots \quad \ldots \quad (1)$$

Determining the period of oscillation T of the rod, AB (or CD), the Young's modulus of the material of the wire, Y, can be obtained.

(ii) **Determination of 'n'**: The rod AB is fixed in a clamp so that the other rod CD and the wire EF form a torsion pendulum (Fig. 64). Now, by setting the rod CD in torsional oscillations and determining its period of oscillation, T_1, the modulus of rigidity of the wire, n is determined from the relation,

$$n = \frac{8\pi I l}{T_1^2 a^4} \quad \ldots \quad \ldots \quad (2)$$

(iii) **Determination of σ**: The three constants Y, n and σ for any material are related to one another by the relation, $Y = 2n(1+\sigma)$ (Art. 69).

$$\text{or} \quad \sigma = \frac{Y}{2n} - 1 \quad \ldots \quad \ldots \quad (3)$$

Substituting for Y and n in (3), we get

$$\sigma = \left(\frac{1}{2} \cdot \frac{8\pi I l}{T^2 a^4} \times \frac{T_1^2 a^4}{8\pi I l}\right) - 1 = \left(\frac{T_1^2}{2T^2} - 1\right) \quad (4)$$

Determining T and T_1 experimentally, σ is calculated.

I-FORM GIRDERS

94. A girder supported at its two ends as on the opposite walls of a room, bends under its own weight and, or, under the load placed above it. The mid-point of the beam is consequently depressed.

A little consideration of the deformation produced in the girder shows that *the parts of the beam, above the neutral surface, contract while those below the neutral surface extend.* The strain of compression or extension in any part is proportional to its distance from the neutral surface. The stresses on the different parts of the beam are, therefore, not equal but vary from a minimum value at the neutral surface to a maximum value at the top and bottom. This suggests that for any given area of cross-section, it would be more economical to have a larger fraction of the area towards the top and the bottom than near the neutral surface. The maximum advantage would be obtained by having the area of cross-section almost coinsiding with the top and bottom surfaces entirely and, a cross-section in the form I satisfies this condition best. Therefore, girders are usually made in the form I for economy.

Fig. 65

Further, the depression e of the mid-point of any girder varies in accordance with the relation, $e = \frac{Mgl^3}{4bd^3Y}$. It is desirable to have as small a depression, e, as possible. Obviously, this can be achieved by providing an increase in the values of b, d or Y.

Since 'd' occurs in the third power, increase in its value produces a better desired effect than an equal increase in 'b' would produce. The vertical length of the section I is therefore increased suitably. *Steel is commonly used* in the manufacture of these I-form girders since it has a large value for Young's modulus, Y.

The commonly available girders are therefore of I-form and they are made of steel having suitable values for the depth, d, the breadth b of the horizontal flanges and thickness t of the vertical section.

Examples

1. Calculate the work done in stretching an elastic wire of length 4·2 metres and diameter 1·4 mm. by 2·1 mm.

($Y = 20 \times 10^{11}$ dynes / sq. cm.)

Work done per unit volume = $\frac{1}{2}$ stress $\times$ strain.

$$\text{Strain} = \frac{\text{Increase in length}}{\text{Original length}} = \frac{0.21}{420} = \frac{1}{2000}$$

Stress = Young's modulus $\times$ strain

$$= 20 \times 10^{11} \times \tfrac{1}{2000} = 10^9 \text{ dynes / sq. cm.}$$

Work done per unit volume

$$= \tfrac{1}{2} \times 10^9 \times \frac{1}{2000} = \tfrac{1}{4} \times 10^6 \text{ ergs. / c.c.}$$

Volume of the wire

$$= \pi \times (0.07)^2 \times 420 \text{ c. c.}$$

Hence total work done

$$= \tfrac{1}{4} \times 10^6 \times \pi \times (0.07)^2 \times 420 = 1.616 \times 10^6 \text{ ergs.}$$

2. A uniform bar of length 150 cm. is clamped horizontally at one end and a load of 100 gm. attached at its free end. Calculate the depression at the loaded end if $Y = 9.78 \times 10^{11}$ dynes/sq.cm. and the bar is of breadth 2.4 cm. and thickness 0.5 cm.

Find also the period of vibration if the bar is set in transverse vibrations (neglecting the effect of the mass of the bar).

The depression at the end of the bar fixed at one end and loaded at the other, $e = \dfrac{4Mgl^3}{bd^3Y}$.

Load suspended, M = 100 gm.
Length of the bar, l = 150 cm.
Breadth of the bar, b = 2·4 cm.
Thickness of the bar d = 0·5 cm.
Young's modulus, Y = 9.78×10^{11} dynes/sq. cm.

$$\text{Depression, } e = \frac{4 \times 100 \times 980 \times 150^3}{2.4 \times (0.5)^3 \times 9.78 \times 10^{11}} = 4.51 \text{ cm.}$$

Period of vibration T when set in transverse vibration is given by

$$T = 2\pi\sqrt{\frac{Ml^3}{3\,YI}} = 2\pi\sqrt{\frac{4Ml^3}{bd^3Y}}$$

$$= 2\pi\sqrt{\frac{4 \times 100 \times 150^3}{2.4 \times (0.5)^3 \times 9.78 \times 10^{11}}} = 0.426 \text{ sec.}$$

Exercises

1. (a) Define 'stress', 'strain' and 'modulus of elasticity.'

(b) Describe the behaviour of a tensile wire fixed at one end and subjected to a gradually increasing load at the other end, explaining clearly the terms 'range of perfect elasticity', 'yield point' and 'permanent set.'

(c) Explain what you understand by 'elastic after-effect' and 'elastic fatigue'.

2. Define Young's modulus.

Describe an accurate method of determining the Young's modulus of a uniform metal wire.

A steel wire, 1 sq. mm. in section and 5 metres long, is stretched by 0·5 cm. Find the stretching force.
(Y of steel = $18{\cdot}5 \times 10^{11}$ dynes/sq. cm.)

[*Ans.* $18{\cdot}5 \times 10^{6}$ dynes]

3. A wire 3 metres long and 0·3 mm. in diameter elongates 1·32 mm. when stretched by a force of 600 gm. weight. Find (a) Young's modulus and (b) the energy stored up in the wire.

[*Ans.* *(a)* $19{\cdot}0 \times 10^{11}$ dynes per sq. cm.
(b) $1{\cdot}830 \times 10^{5}$ ergs. per c. c.]

4. Show that the work done per unit volume in producing a shearing strain is [$\frac{1}{2}$ (Stress) $\times$ (Strain)].

5. Describe with relevant theory an experiment to determine σ of the material of a wire by Searle's method.

6. A steel bar of rectangular cross-section (2.5 cm. in breadth and 1.0 cm. in depth and 100 cm. in length) is supported at its ends and a load of 5 kg. is applied at its mid-point. Calculate the depression at the mid-point. (Y = $19{\cdot}5 \times 10^{11}$ dynes/sq. cm.)

[*Ans.* 0·2519 cm.]

7. A gold wire 0·32 mm. diameter elongates 1 mm. when stretched by a force of 330 gm. weight, and twists through a radian when a twisting couple of 145 dyne-cm. is applied at the ends. Find σ for gold. [*Ans.* 0·428]

8. Show that, for a cantilever of length, l, and carrying a load of weight, W, at its free end, depression y at a distance x from the fixed end is given by $y = \frac{W}{YI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$.

9. Given that A, B, C are three points which are not collinear, explain how they may be joined by an arc of a circle, given a light wooden lath.

10. A uniform wire of length 450 cm. extends by ·36 mm. when stretched by a force of 500 gm. weight. Calculate the Young's modulus of the material of the wire if it has a diameter of 0·62 mm. Find also the energy stored in the wire.

[*Ans.* 20·29 × 10^{11} dynes/sq. cm. and 6492 ergs/c.c.]

11. A brass wire of length 450 cm. and of cross-section 1 sq. mm. is fixed at one end and a load of 10 kg. attached at the other end. Find Y if it extends by 0·45 cm. [*Ans.* 9·8 × 10^{11} dynes/sq. cm.]

12. An elastic bar of length of 1 metre and circular cross-section of diameter 1 cm. is clamped horizontally at one end and a load of 500 gm. is suspended at the free end. Find the depression at the loaded end if the Young's modulus of the material of the bar is 19·8 × 10^{11} dynes/sq. cm.

If the bar is now set in transverse oscillations, find its period.

[*Ans.* 1·681 cm.; 0·26 sec.]

13. A uniform metal bar one metre long is placed symmetrically on two knife-edges separated by a distance of 64·2 cm. When two equal weights of 500 gm. each are suspended from points 5 cm. from the two ends respectively, the mid-point of the bar is elevated by 0·76 mm. Calculate the Young's modulus of the material of the bar if the width of the bar is 2·2 cm. and its thickness 0·62 cm. [*Ans.* Y = 9·81 × 10^{11} dynes/sq. cm.]

14. Find the depression at the mid-point of a bar supported on two knife-edges separated by 65 cm. when loaded at the mid-point with a weight of 400 gm. (b = 1·95 cm.; and d = 0·45 cm., and Y = 9·65 × 10^{11} dynes/sq. cm. [*Ans.* 1·57 mm.]

15. Two beams, one of circular cross-section and the other of square cross-section are used successively in bending experiments. If

they are of same material, length and weight, compare the loads required to produce equal depressions. [*Ans.* $3 : \pi$]

16. Describe, with necessary theory, how 'n' of the material of a rod is determined by the static method.

A disc of radius 10 cm. and mass 460 gm. is suspended in a horizontal plane by means of a wire of length 120 cm. and diameter 1·2 mm. Find the modulus of rigidity of the wire if the disc is found to make 20 oscillations in 50·5 sec.

[*Ans.* $8{\cdot}387 \times 10^{11}$ dynes/sq. cm.]

17. A metal rod of length 46 cm. and of radius 1·5 cm. is suspended to lie in a horizontal plane by means of a wire of length 148 cm. and of diameter 0·75 mm. If the period of vibration of the bar is 10·32 sec., find the modulus of rigidity of the wire given that the rod weighs 150 gm. [*Ans.* $4{\cdot}268 \times 10^{11}$ dynes/sq.cm.]

18. Calculate Poisson's ratio of silver from the following data: Modulus of rigidity for silver = $2{\cdot}9 \times 10^{11}$ dynes/sq cm., Young's modulus for silver = $7{\cdot}9 \times 10^{11}$ dynes/sq.cm. [*Ans.* 0·362]

19. Find the modulus of rigidity for copper given that the Young's modulus for copper is $12{\cdot}5 \times 10^{11}$ dynes/sq.cm. and its Poisson's ratio 0·26. [*Ans.* $4{\cdot}96 \times 10^{11}$ dynes/sq.cm.]

20. Derive, from fundamentals, the relation between Young's modulus, rigidity modulus and bulk modulus of an isotropic material. What is Poisson's ratio? [*A. U. March 1961*]

21. A bar of rectangular section, fixed at one end and free at the other, is loaded at the free end. Derive the expression for the depression at the free end. Explain how this expression is used to find the Young's modulus of the material of a bar.

[*A U. Sept., 1961*]

22. What is cantilever? Derive an expression for the bending of a rectangular beam fixed at one end and loaded at the other, weight of the beam being neglected. How is the Young's modulus of its material determined? [*A. U. April 1962*]

23. Find an expression for the torsional rigidity of the material of a cylinder fixed at the upper end and twisted at the lower end by a couple acting perpendicular to its length.

Define the terms (a) bending moment; (b) neutral surface of a bent beam and (c) flexural rigidity. [*A. U. Sept., 1962*]

24. Obtain the relation between the elastic moduli of a substance and discuss the limits of Poisson's ratio. Indicate how the Poisson's ratio of a substance in the form of a wire is determined. [*A. U. April 1963*]

25. Define the various moduli of elasticity of an elastic solid. Describe with necessary theory a method of determining the Young's modulus of the material of a rod of circular section by bending it. [*M. U. March 1963*]

26. How is the Young's modulus of the material of a bar determined from the study of the vibrations of rod clamped at one end and loaded at the other? Derive the formula used in this determination.

A steel bar 30 cm. long, 2 cm. broad and 0·2 cm. thick is clamped at one end and loaded at the other with a mass of 100 gm. Calculate the period of vibrations of the bar neglecting the effect of the mass of the bar. [*Ans. 1·154 sec.*] [*A. U. Sept, 1963*]

27. Explain 'Young's modulus', 'rigidity modulus' and 'Poisson's ratio' of a material.

Derive a relation connecting them.

Describe how the Poisson's ratio of the material of a short wire can be determined. [*A. U. April 1964*]

28. Find an expression for the couple required to twist a circular cylinder.

A circular disc of mass 200 gm. and radius 5 cm. is fixed perpendicularly to the end of a wire of length 30 cm. and diameter 1 mm. Find the rigidity modulus of the wire if the period of torsional oscillations is 1 sec. [*M. U. April, 1964*]

[*Ans.* $3{\cdot}014 \times 10^{11}$ *dynes/sq. cm.*]

29. Define 'bending moment' and 'neutral axis' of a bent beam. Derive relation between the bending moment and the curvature of the neutral axis. Describe the method of determining the Young's modulus of the material of beam by bending it uniformly. [*A. U. Sept., 1964*]

30. Describe with necessary theory Searle's method of finding n, y and σ. Show that $\sigma < \frac{1}{2}$. [*M. U. Sept., 1964*]

31. Find the relation between the bending moment and the curvature of the neutral axis. Describe a method of finding the Young's modulus by the uniform bending of a bar; deduce the formula to be used. [*M. U. April 1965*]

32. Define 'stress' and 'strain' and 'Young's modulus'. Describe a method of measuring the Young's modulus of a beam of rectangular cross-section. Derive the formula you use. [*A. U. April 1965*]

33. Describe how you would measure the modulus of rigidity of a material in the form of a wire and give the theory of the method. [*A. U. Sept., 1965*]

34. Define the terms (a) bending moment, (b) neutral axis. Describe how the Young's modulus may be determined for material of a bar by bending it uniformly. [*A. U. April 1966*]

35. Obtain the relation between the bending moment and the radius of curvature of a uniform bar bent into an arc of small curvature.

A cantilever of rectangular section has a length of 50 cm. Its breadth is 3 cm. and thickness 0·6 cm. A weight of 1 kg. is attached at the free end. The depression produced is 0·42 cm. Calculate the Young's modulus of the material of the bar. ($g = 980$ cm./sec^2).

[*Ans.* $18{\cdot}01 \times 10^{11}$ *dynes / sq. cm.*] *A. U. April 1965*]

36. Define 'Young's modulus' and 'Poisson's ratio'. Obtain expressions for them in terms of the moduli of bulk and rigidity. [*A. U. Sept., 1965*]

37. Define 'Young's modulus', 'rigidity modulus' and 'bulk modulus'. Show that $Y = 9nk / (3k+n)$ where Y is the Young's modulus, n the rigidity modulus and k the bulk modulus. [*A. U. March 1966*]

38. Describe the torsion pendulum method of finding the rigidity modulus; deduce the formula employed when the moment of inertia of the holder is to be eliminated. [*M. U. April 1966*]

39. Write notes on :

(*a*) Elastic fatigue and elastic limit. [*M. U. Sept., 1963*]

(*b*) Single cantilever. [*M. U. Sept., 1964*]

(*c*) Poisson's ratio. [*M. U. April 1966*]

(*d*) I-form girders [*A. U. April 1968*]

40. Find an expression for the period of oscillation of a torsion pendulum.

Show how the rigidity modulus of the material of a wire may be determined by the method of oscillations. [*A. U. April 1965*]

41. Derive an expression for the period of oscillation of a torsion pendulum.

The period of oscillation of torsional vibrations of a circular disc of radius 5 cm. and mass 500 gm., suspended by a wire of length 1 metre and diameter 1 mm. is found to be 1·7 sec. (The axis of suspension is normal to the plane of the disc). Evaluate the rigidity modulus of the material of the wire. [*Ans.* $9{\cdot}728 \times 10^{11}$ *dynes/sq. cm.*] [*A. U. March 1966*]

42. Describe, with the relevant theory, how you would determine Young's modulus of a material given in the form of a rectangular bar. [*A. U. Oct., 1966*]

43. Derive an expression for the couple required to twist a cylinder fixed at one end, through an angle θ about its axis.

A wire of length 1 metre and diameter 1 mm. is fixed at one end and twisted at the other end through an angle of 70°, by applying a couple of value 100,000 dyne-cm. Evaluate the rigidity modulus of the wire. [*Ans.* $8{\cdot}281 \times 10^{11}$ *degrees/sq.cm.*] [*A. U. Oct., 1966*]]

44. Explain the terms " Young's modulus ", rigidity modulus " and " bulk modulus ".

Derive the relation connecting these three. Calculate the work done in producing an extension of 2 mm. in a steel wire of length 2 metres and diameter 1 mm. Y of steel = 20×10^{11} degrees per sq. cm. [*Ans. 0·157 Joules*] [*A. U. April 1967*]

45. Obtain an expression for the couple required to twist the end of a wire of circular cross-section, the other end of which is fixed, through a given angle.

Hence obtain an expression for the period of oscillation of a torsion pendulum. *[A. U. April 1967]*

46. Describe the torsion pendulum method of finding the rigidity modulus. Deduce the formula employed when the moment of inertia of the holder is to be eliminated.

A circular disc of mass 250 gm. and radius 5 cm. is fixed perpendicularly to the end of a wire of length 30 cm. and diameter 1 mm. Find the rigidity modulus of the wire if the period of torsional oscillations is 1·05 sec. [*Ans. 3.418* $\times 10^{11}$ *dynes/sq. cm.*]

[*A. U. Oct., 1967*]

47. Describe the torsion pendulum method of determining the rigidity modulus of the material of a wire.

A sphere of mass 800 gm. and radius 3 cm. is suspended from a wire of length 100 cm. and radius 0·5 mm. If the period of torsional oscillations is 1·23 sec, calculate the rigidity modulus of the material of the wire. [*Ans. 9·41* $\times 10^{11}$ *dynes/ sq. cm.*] [*A U. April 1968*]

48. A light beam of rectangular cross-section is fixed at one end and loaded at the other end. If the free end of the beam is pressed down a little and released show that the beam executes simple harmonic oscillations and obtain an expression for the period of oscillations. [*A. U. April 1968*]

49. Define the terms 'bending moment' and 'neutral axis.'

Describe with relevant theory an experiment to determine the Young's modulus of the material of a bar by uniform bending.

[A. U. Oetober 1968]

50. Describe with necessary calculations Searle's method of determining the Young's modulus, rigidity modulus and Poisson's ratio of the material of a wire.

A gold wire 0·8 mm. in diameter elongates by 2 mm. when stretched by a force of 82·5 gm. weight and twists through an angle of 60° when a twisting couple of 150 dyne-cm. is applied at the ends. Calculate the Poisson's ratio for gold.

[*Ans. 0·129*] [*A. U. Oct., 1968*]

Chapter V

GRAVITATION

95. Great and revolutionary contributions were made by some outstanding personalities of the 16th and 17th centuries which laid the foundations for a proper understanding of Nature.

Before Copernicus, it was generally believed that the earth was the centre of the universe and that the sun, the moon, the planets and the stars moved round it in circular orbits.

Copernicus, in 1543 A. D., challenged this belief and advanced his *Heliocentric Theory* according to which the sun is the centre of the solar system and all the planets, including the earth, rotate round the sun. His theory was not accepted in the beginning but gradually, on account of the developments that it led to and the confirmations obtained for it, it came to be accepted as a fundamental truth.

Up to the time of Galileo, it was thought that bodies let fall to the ground from a height do not fall at the same rate: that heavier bodies fall more quickly and lighter ones, more slowly. From the leaning tower of Pisa, **Galileo** demonstrated that all bodies, light or heavy, fall to the earth at the same rate. He thus showed that the attraction, that the earth has for a body, i.e., its weight, is proportional to its mass. Galileo also discovered the laws of motion which were later clarified by Newton and are now called **Newton's laws of motion.**

According to these laws, a body moving with uniform velocity in a straight line does not require a force to maintain that motion. Any change in velocity, either in magnitude or in direction, requires a force to produce it and the rate of change of velocity, i.e., acceleration, is proportional to the force that produces it and takes place in the direction in which the force acts. A heavenly body is not subject to forces of resistance due to friction or air, as a body on the surface of the earth is, and therefore should be moving in a straight line, if it is not acted on by any force. If it moves along a circle, there must therefore be a force that deflects it constantly at right angles to its direction of motion.

It was these ideas that led **Newton** to think of a force directed towards the centre of the circular orbit. Though **Copernicus** stated that the sun is at the centre of the solar system, the explanation of the observations of the movements of the planets was not entirely satisfactory. It was because the explanation was based on the assumption that the orbit of heavenly body is a circle, the centre of which is either fixed or moves on another circle and no other geometrical figure was thought of for the orbit than a circle.

Based on the observations of **Tycho Brahe, Kepler** enunciated three laws which were found to satisfy planetary motion very closely and they are called **Kepler's laws of planetary motion.**

96. Kepler's Laws

1. Each planet moves around the sun in an elliptical orbit, the sun being at one of the foci of the ellipse (Fig. 66).

2. The radius vector joining the sun to a planet sweeps out equal areas in equal intervals of time, (i.e., the areal velocity of a planet round the sun is constant).

3. The square of the period of revolution of a planet round the sun is proportional to the cube of the sem-major axis of the ellipse.

Kepler was, however, unable to offer any explanation for his laws as he and his contemporaries were under the impression that force acting on a planet is in the direction of its motion.

97. From the second law of Kepler, Newton deduced that the only force that acts on a planet is along the line joining the planet to the sun and that there is no transverse acceleration. From the third law Newton deduced that the magnitude of the force is inversely proportional to the square of the distance between the planet and the sun. This may be shown in the case of circular orbits as follows :

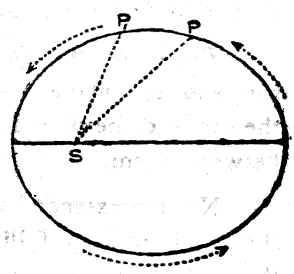


Fig. 66

Let m_1 and m_2 be the masses of two planets ;

r_1 and r_2 the radii of their circular orbits ;

and T_1 and T_2 the periods of revolutions round the sun. The centrifugal force acting on the first planet,

$$F_1 = m_1 r_1 \left(\frac{2\pi}{T_1}\right)^2.$$

Similarly, the centrifugal force acting on the second planet,

$$F_2 = m_2 r_2 \left(\frac{2\pi}{T_2}\right)^2.$$

Now according to Newton, the only force acting on the planet due to the sun is directed towards the latter and it is equal to the centrifugal force.

$\therefore$ F_1 and F_2 will be equal to the forces of attraction exerted by the sun on the two planets respectively.

$$\text{Now, } \frac{F_1}{F_2} = \frac{m_1 r_1}{m_2 r_2} \cdot \left(\frac{T_2}{T_1}\right)^2.$$

$$\text{But by Kepler's third law, } \left(\frac{T_2}{T_1}\right)^2 = \left(\frac{r_2}{r_1}\right)^3.$$

$$\therefore \frac{F_1}{F_2} = \frac{m_1 r_1}{m_2 r_2}\left(\frac{r_2}{r_1}\right)^3 = \frac{m_1}{m_2} \times \frac{r_2^2}{r_1^2}.$$

This shows that the force F varies directly as the mass of the planet and inversely as the square of the distance of the planet from the sun (Inverse square law).

According to the *third law of motion*, the planet also exercises attraction on the sun and this force must be equal and opposite to F and it should vary directly as the mass of the sun.

Therefore the force of gravitational attraction between the sun and the planet varies directly as the product of the masses of the sun and the planet and inversely as the square of the distance between them.

Newton extended the idea to the rotation of the moon round the earth and to falling bodies. He thought that *the force which the earth exercises on a falling body near its surface towards it also is responsible for the centripetal force on the moon which maintains* the moon in its orbit. His idea was confirmed by actual calculations (Art. 99) and the inverse square law has therefore become a Universal Law of Gravitation.

98. The Law of Universal Gravitation

Every particle of matter attracts every other particle of matter with a force which varies directly as the product of their masses and inversely as the square of the distance between them.

Thus, if F be the force of attraction between two particles of matter of masses m_1 and m_2 and d, the distance between them, then $F \propto m_1 \cdot m_2$

and $F \propto \frac{1}{d^2}$

i.e., $F \propto \frac{m_1 \cdot m_2}{d^2}$ or $F = G. \frac{m_1 m_2}{d^2}$

where G is a constant called the **Constant of Gravitation.**

The Gravitational constant can be defined as *the force of attraction between two particles of matter, each of unit mass, which are separated from each other by unit distance.*

99. The Moon and the falling body

The law may be applied to the cases of a falling body and the rotation of the moon and its validity tested as was done by Newton.

The weight of a body = mg (∵ the weight is proportional to the mass as shown by Galileo)

If E be the mass of the earth and r its radius, it follows from the Law of Universal Gravitation that the force of gravitational attraction on the body near the earth = $G. \frac{Em}{r^2}$. But this is equal to the weight of the body = mg.

$$\therefore G. \frac{Em}{r^2} = mg \quad \text{or} \quad g = \frac{GE}{r^2} \qquad \cdots \qquad \cdots \qquad (1)$$

If M be the mass of the moon and R its distance from the earth, i.e., the radius of the moon's orbit, the force of attraction, between the earth and the moon, according to the Law of Gravitation

$$= G. \frac{EM}{R^2} \qquad \cdots \qquad \cdots \qquad (2)$$

If ω be the angular velocity of the moon, the centripetal force required to maintain the motion of the moon in its orbit,

$$= M\omega^2 R \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

According to Newton, this centripetal force is supplied by the Gravitational attraction between the earth and the moon,

$$\text{i.e., } \frac{GEM}{R^2} = M\omega^2 R.$$

or, the acceleration of the moon towards the earth,

$$\omega^2 R = \frac{GE}{R^2} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$\text{From (1) and (4), } \frac{\omega^2 R}{g} = \frac{r^2}{R^2} \quad \ldots \quad \ldots \quad (5)$$

i. e., the accelerations are inversely proportional to the squares of their respective distances from the earth.

Values may now be substituted to verify the relation (5).

The moon revolves round the earth once in 27·3 days, i.e., its period of revolution, T ($= 27{\cdot}3 \times 24 \times 60 \times 60$) sec.

$$= 2{\cdot}359 \times 10^6 \text{ sec.}$$

$\therefore$ The angular velocity of the moon,

$$\omega = 2\pi/2{\cdot}359 \times 10^6 = 2{\cdot}663 \times 10^{-6} \text{ rad./sec.}$$

The distance, R, of the moon from the earth,

$$= 2{,}40{,}000 \text{ miles} = 2{,}40{,}000 \times 5{,}280 \text{ ft.} = 1{\cdot}267 \times 10^9 \text{ ft.}$$

$\therefore$ The (normal) acceleration of the moon towards the earth,

$$\omega^2 R = (2{\cdot}663 \times 10^{-6})^2 \times 1{\cdot}267 \times 10^9 = 8{\cdot}982 \times 10^{-3}.$$

Acceleration due to gravity, g, on a falling body $= 32{\cdot}2$ ft./sec.^{2}

$$\therefore \omega^2 R/g = 0{\cdot}0002789 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Radius of the earth = 4,000 miles.

$$\therefore \frac{r^2}{R^2} = \left(\frac{4{,}000}{2{,}40{,}000}\right)^2 = \left(\frac{1}{60}\right)^2 = 0{\cdot}0002777 \quad \ldots \quad (7)$$

It is seen that there is agreement in the values obtained in (6) and (7) which establishes the truth of the law.

100. Gravitational field and intensity

According to the law of gravitation every material particle (A) exerts a force of attraction on any other material particle (B) that

may be situated in the space surrounding it. *The space surrounding the particle A in which gravitational force of attraction is felt is called the* **gravitational field of A.**

The intensity or strength of the gravitational field at any point is defined as the force in dynes experienced by a particle of unit mass when placed at that point. Thus the intensity at a point, distant x from a body of mass, M, according to the law of gravition,

$$= G. \frac{M \cdot 1}{x^2} = \frac{GM}{x^2}.$$

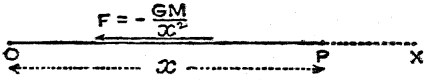


Fig. 67

It would be better if signs also are taken into consideration. If the mass M be supposed to be situated at the origin of co-ordinates, O, the force of attraction on unit mass at any point P is directed towards O, and therefore it is negative when x is positive. On the assumption that x is positive, we may write that

$$F = - \frac{GM}{x^2} \quad \ldots\ldots \quad \ldots\ldots \quad (1)$$

101. Gravitational potential : If the unit mass be displaced by a distance x, further from the mass M, a certain amount of work has to be done against the gravitational intensity. Now this work will be stored up as potential energy in the unit mass.

The work done, when unit mass is moved from a point A to a point B against the intensity of the gravitational field, is called the difference in potential between the two points.

Thus when P and Q are separated by a small distance dx, the increase of the potential, dv

= work done in moving unit mass from P to Q
= Force on unit mass $\times$ displacement
= Intensity of the field, F $\times$ displacement, dx
= $-$ F $\times$ dx

(the negative sign indicates that the displacement is opposite in direction to F).

102. Potential due to a point mass : If a mass, m, is situated at O, and a unit mass be placed at P, the force on the unit mass, $F = -\frac{Gm}{x^2}$.

If the unit mass be given a displacement dx,

then $dv = -F \times dx = \frac{Gm}{x^2} . dx$.

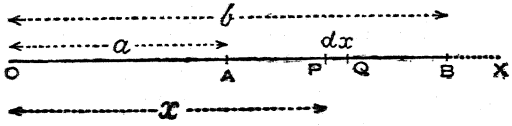


Fig. 68

If the unit mass be displaced from A to B (where OA = a and OB = b), the difference of potential =

$$V_B - V_A = \int_{V_A}^{V_B} dv = \int_a^b \frac{Gm}{x^2} dx = -Gm \left[\frac{1}{x}\right]_a^b$$

$$= -Gm \left(\frac{1}{b} - \frac{1}{a}\right)$$

$$\text{or } V_B - V_A = Gm \left(\frac{1}{a} - \frac{1}{b}\right)$$

If the point B be at infinity, $b = \infty$ and the work done in moving unit mass from A to infinity

$= G.m. \left(\frac{1}{a} = \frac{Gm}{a}\right)$, a quantity which is positive.

Now the intensity of the gravitational field at a point which is at infinity from O, is equal to zero and the work done in moving a unit mass through a small distance at infinity is therefore zero. Therefore the potential at all points at infinity is the same. It has become the practice to consider this potential to be zero and to define the potential at a point as the work done in moving unit mass from infinity to the point.

Since the work done in moving unit mass from A to infinity is equal to $\frac{Gm}{a}$, the work done in moving unit mass from infinity to

the point A will be equal to $-\frac{Gm}{a}$ and therefore the potential at a point A, distant 'a', from a mass $m = -\frac{Gm}{a}$.

Thus, gravitational potential has the maximum value of zero at infinity and decreases as the distance is decreased, being always negative.

103. Potential due to a spherical shell

Consider a uniform hollow spherical shell of radius 'a'. Let ρ be the mass per unit area of the surface of the shell.

(a) Let P be any point outside the shell which has for its centre, the point O. Join OP and let OP = x.

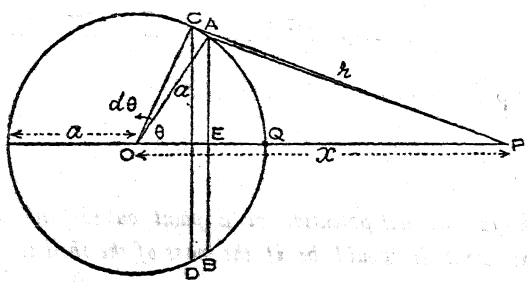


Fig. 69

Consider a thin slice of the shell contained between two planes AB and CD drawn close to each other at right-angles to OP. Join O and C, O and A and A and P.

Let $\angle AOP = \theta$ and $\angle AOC = d\theta$.
Now, AE = the radius of the slice $= a \sin \theta$.
Circumference of the slice $= 2\pi a \sin \theta$.
Width of the slice (i.e., CA) $= a.\, d\theta$.
Surface area of the slice $= (2\pi a \sin \theta)\ a.\, d\theta$.
$\therefore$ Mass of the slice $= 2\pi a^2.\ \rho.\ \sin \theta.\ d\theta$.
Let PA = r.
Then, potential due to the slice at P

$$= -G\,\frac{2\pi a^2 \rho \sin \theta\, d\theta}{r} \quad \ldots \quad (1)$$

In the $\triangle$ AOP, $r^2 = a^2 + x^2 - 2ax \cos\theta$.

$\therefore\ 2r\,dr = 2ax \sin\theta\, d\theta$

or $$\frac{dr}{x} = \frac{a \sin\theta\, d\theta}{r} \quad \ldots\ \ldots \quad (2)$$

Substituting this result in (1),

the potential at P due to the slice $= -G\frac{2\pi a\rho}{x} \cdot dr$ .... (3)

$\therefore$ Potential due to the spherical shell

$$= -G\frac{2\pi a\rho}{x}\int_{x-a}^{x+a} dr$$

$$= -G.\frac{2\pi a\rho}{x} \cdot 2a = -G\frac{4\pi a^2 \cdot \rho}{x}.$$

But $4\pi a^2\rho$ = the mass of the shell, M.

Hence the potential at P due to the shell $= -\frac{GM}{x}$.

Thus *the gravitational potential at a point outside the spherical shell is the same as what it would be if the mass of the shell is concentrated at its centre.*

(b) **Point on the shell**: Let P be shifted to Q so that it lies on the surface and OQ $= x = a$.

From the expression (3), the potential due to the slice

$$= -G.\frac{2\pi a\rho}{x} \cdot dr = -G.2\pi\rho \cdot dr \ (\because x = a)$$

Potential due to the shell $= -G.2\pi\rho\int_0^{2a} dr$

$$= -G.4\pi a\rho = -\frac{G.4\pi a^2\rho}{a} = -\frac{GM}{a}.$$

The potential at any point on the surface of the shell is also the same as what it would be if the mass is concentrated at the centre.

(c) **Point inside the shell :** In this case the limits for 'dr' are $(a + x)$ and $(a - x)$ so that the potential due to the entire shell

$$= -\text{G}.\frac{2\pi a\rho}{x}\int_{a-x}^{a+x} dr = -\text{G}\,.\,4\pi a\rho$$

$$= -\text{G}.\frac{2\pi a^2\rho}{a} = -\frac{\text{GM}}{a}.$$

Thus the potential at any paint inside the shell is of the same value throughout the shell and is equal to the value on its surface.

104. Potential due to a solid sphere (of radius a)

A solid sphere may be imagined to be made up of a number of infinitely thin concentric spherical shells each having the same density throughout and the radii of which vary from 0 to a.

(a) **Point P outside the sphere :** Each one of the shells produces a potential at the point P outside the shell as if its entire mass is concentrated at the centre O.

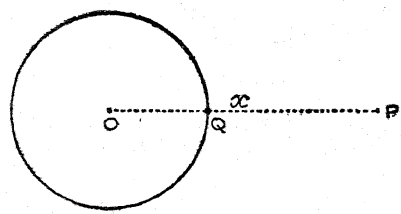


Fig. 70

Thus if m be the mass of one such shell, the potential due to the shell at P $= -\dfrac{Gm}{x}$.

$\therefore$ Potential due to the whole sphere

$$= -\text{M}\frac{Gm}{x} = -\frac{\text{G}}{x}\text{M}\,m$$

$$= -\frac{\text{GM}}{x}$$ where M is the mass of the sphere.

(b) **Point P on the surface :** In this case also, each one of the shells produces a potential at the point as if its mass is concentrated at its centre.

Hence the potential at a point on the surface $= -\dfrac{\text{GM}}{a}$.

(c) **Point P inside the sphere :** Let the point P be situated at a distance x from the centre of a solid sphere of radius 'a'.

Imagine a sphere to be described with centre 'O' and radius 'x'. The sphere divides the given solid sphere into two parts :

(i) a solid sphere of radius x on which lies the point P and (ii) a hollow sphere lying between spheres of radii, a and x.

(i) Now the mass of the inner sphere of radius $x = \frac{4}{3}\pi x^3\rho$ (where ρ is the density of the material of the sphere).

$\therefore v_1$ = Potential due to the sphere at P

$$= -G\,\tfrac{4}{3}\pi x^3\rho / x = -G\,\tfrac{4}{3}\pi x^2\rho \quad \ldots \quad \ldots \quad (1)$$

(ii) Let the hollow sphere be imagined to be divided into a number of concentric shells.

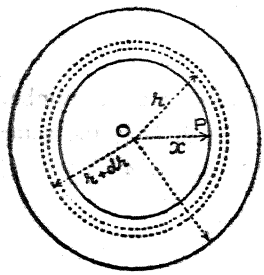


Fig. 71

Let one such shell be of radius r and thickness dr.

Mass of the shell $= 4\pi r^2 dr\cdot\rho$.

Potential due to the shell at P inside = Potential on the surface of the shell of radius, r

$$= -G.\frac{4\pi^2 dr\rho}{r}$$

$$= -G.\,4\pi r dr\,\rho.$$

$\therefore$ if v_2 be the potential due to all the shells outside P,

$$v_2 = -G.\,4\pi\rho\int_x^a r.dr = -G.4\pi\rho\left(\frac{a^2}{2} - \frac{x^2}{2}\right) \quad \ldots (2)$$

$\therefore$ The potential due to the entire sphere,

$$v = v_1 + v_2 = -G.\tfrac{4}{3}\pi x^2\rho - G.\,4\pi\rho\left(\frac{a^2}{2} - \frac{x^2}{2}\right)$$

$$= -G.\,\tfrac{4}{3}\pi\rho\left(x^2 + \frac{3a^2}{2} - \frac{3x^2}{2}\right)$$

$$= -G.\,\tfrac{4}{3}\pi\rho\left(\frac{3a^2 - x^2}{2}\right)$$

But the mass of the sphere, $M = \frac{4}{3}\pi a^3\rho$.

$$\therefore v = -GM\left(\frac{3a^2 - x^2}{2a^3}\right).$$

105. Intensity of Gravitational field

It has been shown in Art. 101 that the intensity of the field, F, is given by $F = -\frac{dv}{dx}$. The intensity, F, in different cases, can therefore be obtained using this relation.

(a) **Spherical shell**

1. Point outside the shell: Potential, $v = -\frac{GM}{x}$

$\therefore$ Intensity, $F = -\frac{d}{dx}\left(-\frac{GM}{x}\right) = -\frac{GM}{x^2}$ ... (1)

(The negative sign indicates that the force is towards the centre of the sphere, O).

2. Point on the surface: Putting $x = a$ in the expression (1) we get the intensity of the field at a point on the surface of the shell to be $F = -\frac{GM}{a^2}$.

3. Point inside the spherical shell

Potential, $v = -\frac{GM}{a}$ which is a constant

$$\therefore F = -\frac{dv}{dx} = 0$$

The intensity of the gravitational field inside the spherical shell is zero.

(b) **Solid sphere**

(i) **Point P outside the sphere**

Potential, $v = -\frac{GM}{x}$.

$$\therefore \text{Intensity, } F = -\frac{GM}{x^2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

(ii) The intensity of the field at P on the surface of the solid sphere is obtained by putting $x = a$ in expression (2) and therefore $F = -\frac{GM}{a^2}$.

Thus, *the intensity of the gravitational field at any point on the surface of or outside a solid sphere varies inversely as the square of the distance from the centre of the sphere.*

(iii) **Point P inside the sphere:** Potential at a point inside the solid sphere at a distance x from the centre O,

$v = -G.M\left(\frac{3a^2 - x^2}{2a^3}\right)$.

$$\therefore \text{ The intensity of the field at P } = -\frac{dv}{dx}$$

$$= -\frac{d}{dx}\left[-\text{GM}.\frac{3a^2-x^2}{2a^3}\right] = -\frac{\text{GM}}{a^3}x.$$

Thus, *the intensity of the field inside a solid sphere varies as x, i.e., as its distance, from the centre of the sphere.*

106. Gravitational constant and the mean density of the earth

The earth is nearly spherical. The attraction exerted by it on any body on its surface or outside it is the same as if its entire mass is concentrated at its centre.

If R be the radius of the earth and D its mean density, its mass $= \frac{4}{3}\pi R^3 D$.

According to the law of gravitation, the force of attraction of the earth on a body of mass m on its surface

$$= G\,\frac{4}{3}\,\frac{\pi R^3 D.m}{R^2} = \frac{4}{3}\,G \cdot \pi RDm \quad \ldots \quad \ldots \quad (1)$$

But the force of attraction of the earth on a body is, by definition, the weight of the body which is equal to mg, g being the acceleration due to gravity.

$$\therefore mg = G\,\tfrac{4}{3}\,\pi RDm$$

$$\text{i.e., } G = \frac{3g}{4\pi RD} \quad \ldots \quad \ldots \quad (2)$$

DETERMINATION OF 'G'

107. Mountain experiments

In the first experiments that were conducted to find the value of the mean density of the earth (D) and the gravitational constant (G), the force of attraction exerted by the earth on the bob of a plumb line was compared with the attraction exerted by a mountain.

In the absence of any mountain, the plumb bob hangs vertically downwards. If the plumb bob be by the side of a mountain, the gravitational force of attraction of the mountain

deflects the plumb line by an angle (θ). If the force due to the mountain is horizontal, we may write, $\tan\theta = \frac{F}{W}$ ... . (1)

where F is the force of attraction due to the mountain and W, the earth's force of attraction, i.e., the weight of the body. Now if M be the mass of the mountain, m mass of the plumb bob, E the mass of the earth, d the distance of the centre of gravity of the mountain from centre of the bob, and R the radius of the earth,

$$F = G\frac{Mm}{d^2} \quad \text{.... ....} \quad (2)$$

$$\text{and } W = mg \left(= G.\frac{Em}{R^2} = G \cdot \tfrac{4}{3}\pi RDm \right) \quad \text{.... ....} \quad (3)$$

$$\tan\theta = \frac{F}{W} = \frac{GMm}{d^2} \Big/ mg$$

$$= \frac{GM}{d^2 g} \quad \text{or} \quad G = \left(\frac{d^2 g \tan\theta}{M} \right).$$

After obtaining the value of G from this relation, D can be obtained making use of the relation $D = \frac{3g}{4\pi RG}$ [from (2) — Art · 106]

(a) The mass of the mountain, M and the position of its centre of gravity were determined by conducting an elaborate survey.

(b) To find θ, a zenith sector was employed. The zenith sector is a telescope mounted near the object glass so as to rotate about a horizontal axis which lies east-west. A plumb line is suspended from the same axis and it moves below over a scale graduated in degrees, minutes and seconds.

Bouguer and **Maskeylene** conducted experiments on these lines o determine the values of D and G.

108. The mountain chosen by Maskeylene was Mt. Schiehallion in Perthshire and it runs east and west. Observations were taken with the zenith sector to the north of the mountain and to the south of the mountain. The telescope was directed towards a star in the zenith position, in both the cases. When the zenith sector was placed to the north of the mountain, the plumb bob would be attracted towards the south. When the zenith sector was to the south of the

mountain, it would be attracted towards north. The difference, β, between the two readings of the plumb line on the scale, which was found to be 55″ was *partly due to the curvature of the earth and partly due to the attraction of the mountain.*

The diflection due to curvature of the earth, α

$$= \frac{\text{the distance between the two places}}{\text{(Radius of curvature of the earth)}}$$

(in radian measure) and it was found to be 43″.

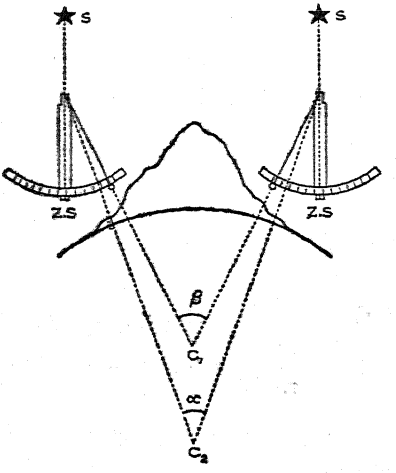


Fig. 72

The deflection, $2\theta\ (=\beta-\alpha)$, due to the attraction of the mountain was therefore 12″ or $\theta = 6''$.

The mean density of the earth D was obtained by Maskeylene to be 4·5 gm. / c.c. and the gravitational constant $7{\cdot}54\times10^{-8}$ C. G. S. units.

This value of G is much higher than the value obtained by more accurate experiments later. The estimation of the mass of the mountain and the location of its centre of gravity involve uncertainties which account for the error.

109. Airy's Mine Experiments

In the mine experiment, the acceleration due to gravity g_1 on the surface of the earth is compared with the value g_2 at the bottom of a mine.

Let E be the mass of the earth, R the radius of the earth, h the depth of the mine, D the mean density of the earth and d the mean density of the crust of the earth of thickness h.

If m be the mass of a body, its weight on the surface of the earth

$$= mg_1 = \text{G.}\frac{Em}{R^2} \text{ or } g_1 = \text{G.}\frac{4}{3}\pi RD \quad \ldots\ (1)$$

When the mass m is taken to a depth h below the surface, the outer shell of the earth of thickness, h, does not exert any attraction

on it and the mass of the sphere of radius $(R-h)$ will be responsible for producing the gravitational attraction.

Now, the volume of the earth in the outer shell of thickness,

$$h = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi (R-h)^3 = 4\pi R^2 h \text{ (approximately).}$$

Mass of the earth in the outer shell

$= 4\pi R^2 hd$ (approximately).

Mass of the earth in the sphere of radius $(R-h)$

$$= \frac{4}{3}\pi R^3 D - 4\pi R^2 hd$$

The relation for attraction according to the law of gravitation

$$= mg_2 = G\left[\frac{\frac{4}{3}\pi R^3 D - 4\pi R^2 hd}{(R-h)^2}\right] m$$

$$\text{or, } g_2 = G\left[\frac{\frac{4}{3}\pi R^3 D - 4\pi R^2 hd}{(R-h)^2}\right] \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\frac{g_1}{g^2} = \left[G.\frac{4}{3}\pi RD\right] \div \left[G.\frac{\frac{4}{3}\pi R^3 D - 4\pi R^2 hd}{(R-h)^2}\right]$$

$$= \frac{G.\frac{4}{3}\pi R^3 D\left(1-\frac{h}{R}\right)^2}{G.\frac{4}{3}\pi R^3 D\left(1-\frac{3hd}{RD}\right)} = 1 - \frac{2h}{R} + \frac{3hd}{RD} \text{ (approx.)}$$

$$\text{i.e., } \frac{g_2 - g_1}{g_2} = \frac{h}{R}\left(2 - \frac{3d}{D}\right) \text{ (approximately)}$$

In 1854, Airy determined with great care, the values of g_1 and g_2 (by pendulum experiments) on the surface of the earth and at the bottom of a mine, the Harton coal pit which is 1250 ft. deep. The mean density d of the earth's crust in the outer shell was obtained by determining the densities of a number of samples of the earth in the shell at different levels. The value of d was found to be 2·5 gm./c.c.

Substituting the values in (1), G was calculated to be $5{\cdot}78 \times 10^{-8}$ C G.S. units and the mean density of the earth D to be 6·5 gm./c.c.

The mean density of the crust is an uncertain factor which cannot be obtained with accuracy and therefore the values for G and D cannot be said to be very accurate.

LABORATORY EXPERIMENTS

110. The masses used in laboratory experiments for studying mutual attraction are necessarily to be smaller than those in the experiments described above and consequently the forces which are to be measured become very small. Refined and sensitive methods have therefore to be employed for measuring them.

Rev. John Mitchell suggested the use of a torsion balance for measuring these forces. He designed an apparatus for this purpose but died before he could use it. The apparatus passed into the hands of Cavendish who used it successfully to determine G and D to a fair degree of accuracy.

111. Cavendish experiment

The apparatus consisted of a light torsion beam PQ of dealwood which was 6 ft. long. It was suspended by means of a silver-copper wire *w* which was 39·25″ long. To strengthen the beam and keep it straight, the ends, P and Q, were connected by silver wires to an upright R fixed at its centre. Two lead balls A and B each of diameter 2″ were suspended from the ends P and Q respectively. The ends of the beam were provided with vernier bits (each of five divisions) which moved over fixed scales, (divided into twentieths of an inch) for noting the deflection of the beam.

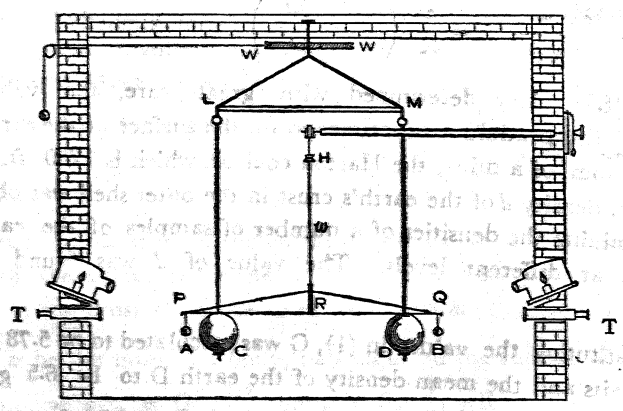


Fig. 73

Two lead spheres C and D each of mass 35 lb. and 12″ in diameter were hung down from the ends of a cross-bar LM by

means of two vertical rods of copper. The four masses A, B, C and D were arranged to lie with their centres in the same horizontal plane. The cross-bar was arranged so that it could be rotated about the same axis as the beam PQ by means of a wheel, WW.

Cavendish took great care to reduce the effect of the air-currents on the equilibrium of the beam PQ. He constructed a special inner chamber in an out-house in his garden for keeping the apparatus. The room was locked and all manipulations were carried on from outside. The cross-bar LM was rotated into the desired positions by means of a rope which was wound over the wheel WW and passed outside. The scales and verniers were illuminated by lamps fitted on the inner side of the walls of the chamber and observed through telescopes, T, T fitted into the walls. For convenience, two short pegs were also driven into the walls to stop the masses, C and D, at equal distances from the centre of the beam PQ.

It is reported that, in spite of all the precautions taken, the beam kept on oscillating on account of air-currents set up inside due to temperature differences in the chamber. Cavendish had therefore to resort to the method of oscillations for determining the position of rest of the beam.

Method of the Experiment: The heavy masses were first arranged so as to have their line of centres at right-angles to the beam. The torsion wire in this position was not twisted. The positions of the verniers on the scales were noted to obtain the position of rest of the beam.

The cross-bar LM was then turned to get the masses C and D near to A and B respectively (as at C_1 and D_1) as shown in Fig. 74, the distances between the masses (C and A) and (D and B) being the same. The masses C and D were on opposite sides of the beam with the lines of centres AC and BD remaining horizontal and perpendicular to the beam. The smaller masses A and B were attracted towards C and D respectively with equal forces so that the beam was subjected to a couple tending to turn it in the clock-wise direction. The positions of the verniers on the scales were noted again. The

difference between these readings and the readings in the position of rest gives the deflection θ_1.

The cross-bar was then swung round to get the mass C near B and the mass D near A (as at C_2 and D_2) as shown in Fig. 74, so as to exert a couple on the beam PQ in the anti-clock-wise direction, the distances between the two masses (A and D) and (B and C) being the same as before. The position of rest of the beam was again noted and the deflection θ_2 obtained. The average of θ_1 and θ_2 gives θ, the twist on the wire due to the deflecting couple.

Theory of the experiment

Let M be the mass of a heavy sphere,

m, the mass of a small sphere,

d, the distance between the centres of two near masses (A and C or B and D),

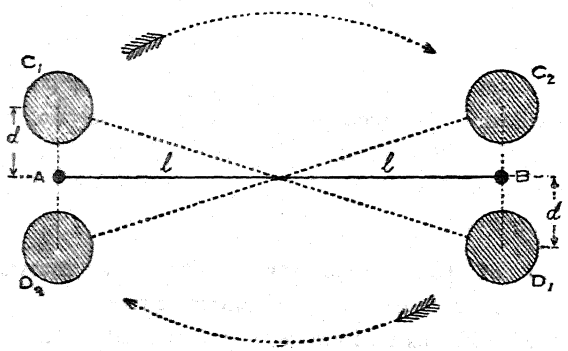


Fig. 74

$2l$, the length of the beam PQ,

μ the moment of the torsional couple per unit twist, and θ, the angle of deflection of the beam.

When the heavier mass M is brought near the smaller mass, m, the latter is acted on by a force due to the gravitational attraction between M and m which is equal to $\left(G \cdot \frac{Mm}{d^2}\right)$. An equal force acts at the other end of the beam in the opposite direction and the two forces exert a couple.

The moment of the couple $= \frac{GMm}{d^2} \times 2l$ .... (1)

and this will tend to twist the suspension wire. When the wire is twisted, it exerts a torsional couple in the opposite direction and if the beam be deflected through an angle θ from its position of rest, the torsional couple brought into play by a twist, $\theta = \mu\theta$ ... (2)

For equilibrium, $G\frac{Mm}{d^2} \times 2l = \mu\theta$ ... (3)

μ is determined separately by determining the periodic time T of the beam when it performs torsional oscillations.

Now, $T = 2\pi\sqrt{\frac{I}{\mu}}$ where I is the moment of inertia of the beam. From this relation, μ is calculated.

Knowing all the other quantities, the gravitational constant G was calculated and from it, the mean density of the earth D, using the relation $D = \frac{3g}{4\pi RG}$.

The values obtained by Cavendish are as follows:

$G = (6.754 \pm 0.041) \times 10^{-8}$ C. G. S. units,

and $D = (5.448 \pm 0.033)$ gm. / c. c.

Cavendish applied corrections for the following sources of error in his experiment:

(a) the attraction of a heavier mass on the remote smaller mass at the other end of the beam,

(b) the attraction of the heavier mass on the beam PQ, and

(c) the attraction of the copper rods supporting the heavier masses on the smaller masses.

The accuracy of his experiment was however limited on account of the following defects:

He used a long torsion beam to reduce the attraction of the heavier masses on the remote smaller ones and also to provide a larger deflecting couple. A thick wire had consequently to be used for suspending the beam, which reduced the value of the deflection θ. Moreover, even the corrections applied were themselves subject to uncertainties. Added to all these, the method of measuring the deflection was not sensitive.

The use of a long beam made the use of a big room necessary which in its turn brought about a certain disadvantage. Convection currents inside the room could not be avoided. In addition, the equilibrium of the beam was affected by the heavy traffic on the road outside.

Boys' method

112. Prof. C. V. Boys considered all the sources of error in the Cavendish experiment and designed a small and compact apparatus in 1895 for determining G.

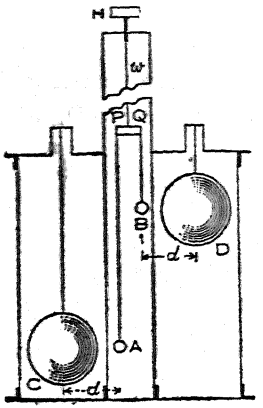


Fig. 75

Prof. Boys developed a technique of drowing very fine and uniform fibres of fused quartz. *He found the quartz fibres to be as strong as steel in their tensile strength but their rigidity was comparatively very low so that a very sensitive torsion balance could be constructed with them, with a great reduction in the size of the balance.*

Boys' apparatus consists of a small mirror strip PQ about 1" long which is suspended from a torsion head by means of a fused quartz fibre. The mirror strip serves as a torsion beam and also for measuring the deflection θ. Two gold spheres A and B, each 0·25" in diameter, are suspended from the ends P and Q. They are arranged to be at different levels, the mass A being 6" lower than B. The gold balls and the suspension are enclosed in a glass chamber of 1·5" diameter. The attracting spheres C and D are of lead and of 4·5" diameter and one of them C arranged to be at the same height as A and the other D at the same height as B.

The experiment is conducted as was done by Cavendish. The angle of deflection, however, is measured by noting the deflection of a beam of light which is reflected from the mirror strip.

The heavier masses C and D are first brought into the position in which they exert maximum couple on the mirror strip, in one direction. The angle of deflection θ is noted. They are then

swung round so that they exert maximum couple again but in the opposite direction (the distance between the attracted and attracting masses being almost the same as before).

As in the case of Cavendish experiment,

if $2l$ be the length of the mirror strip AB,

d, the distance between the attracting and the attracted masses on one side,

M, the mass of a heavy sphere,

m, the mass of a gold sphere,

the deflecting couple $= G \frac{Mm}{d^2} \cdot 2l$ ... (1)

If μ is the torsional couple per unit twist,

torsional couple exerted by the fibre $= \mu \cdot \theta$ ... (2)

For equilibrium, $\mu \cdot \theta = G. \frac{Mm}{d^2} \cdot 2l$ .... (3)

The value of G is calculated from (3). As before μ is calculated from the relation $T = 2\pi \sqrt{\frac{I}{\mu}}$ .... (4)

by observing the periodic time T using the arrangement of the quartz fibre and the mirror strip with gold balls as a torsion pendulum the value obtained for G and D by Boys are

$G = 6.6576 \times 10^{-8}$ C. G. S. units

and D = 5.53 gm./c.c.

The results obtained by him are very accurate.

The advantages of Boys' method are as follows :

1. The use of a quartz fibre has made the apparatus very sensitive and accurate.

2. The dimentions of the apparatus are reduced and a small chamber can therefore be used in which uniformity of temperature can be maintained, thereby preventing convection currents.

3. By arranging the masses at different levels, the effect of the attraction of the heavier mass on the remote smaller mass is very much reduced.

4. By the lamp and scale arrangement, very small deflections can be measured accurately.

113 Poynting's method

In 1893 Poynting actually weighed the attraction between two lead spheres by means of a balance and determined G. A sensitive bullion balance is arranged in an underground room so that it is totally cut off from all extraneous disturbances. All the manipulations are carried out from outside.

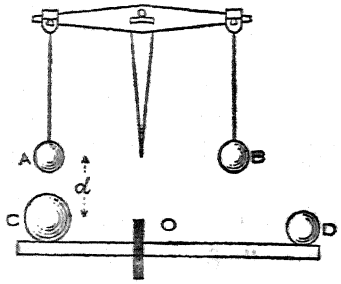


Fig. 76

Two lead spheres A and B each of mass 50 lb. are suspended from the ends of the beam of the balance. The attracting sphere C is also of lead and of mass 350 lb. It is mounted at one end of a turn-table, which is capable of rotating about a vertical axis. The axis passes through the centre of the beam and by turning the turn-table, the mass C can be brought vertically under A or B. To counterpoise the mass C, a smaller mass, D, about half the mass of C is placed on the table at about twice the distance from the axis on the other side. The deflection of the beam is measured by an optical arrangement.

The mass C is arranged under A. The deflection θ_1 of the pointer of the balance on the scale is noted.

Let M be the mass of the attracting sphere, m the mass of the small sphere, and d the distance between them.

$$\text{Force of attraction on the smaller mass} = G.\frac{Mm}{d^2} \quad \ldots \quad (1)$$

If l be the length of the arm of the beam, the moment of the

$$\text{force} = G.\frac{Mm}{d^2} \times l \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

If the corresponding deflection produced be θ_1 and μ the moment of the force required to produce unit deflection,

$$G.\frac{Mm}{d^2} \times l = \mu\theta_1 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The turn-table is then rotated so as to bring the mass C under B. If θ_2 be the deflection now, $G.\frac{Mm}{d^2} = \mu\theta_2$ .... (4)

If θ is the total deflection of the beam produced by changing C from one position to the other, $\theta = \theta_1 + \theta_2$ and from (3) and (4)

$$G.\frac{Mm}{d^2}.\ 2l = \mu\ (\theta_1 + \theta_2) = \mu\ \theta \qquad \ldots\ (5)$$

θ is equal to the difference in the scale readings obtained when C is at A and when it is at B.

To determine μ, a centigram rider is moved on the beam from the centre to a distance x and the corresponding angular deflection ϕ of the beam is noted.

Then $0{\cdot}01\ g \times x = \mu\ \phi$

$$\text{or } \mu = \frac{0{\cdot}01\ g \times x}{\phi} \qquad \ldots \quad \ldots \qquad (6)$$

$$\therefore\ G.\ \frac{2Mml}{d^2} = \frac{0{\cdot}01\ g \times x}{\phi} \cdot \theta \quad [\text{from (5)}]$$

$$\text{or } G = \frac{0{\cdot}01\ gd^2x\theta}{2Mml\ \phi} \qquad \ldots \quad \ldots \qquad (7)$$

The mean values obtained for G and D by Poynting were as follows :

$$G = 6{\cdot}6984 \times 10^{-8}\ \text{C.G.S. units.}$$

and $D = 5{\cdot}4934$ gm. / c.c.

114. Qualities of Gravitation

The force of attraction between any two bodies is not affected by the intervening medium. The force is not affected by the nature, state and structure of the bodies involved but depends entirely on their masses. Bodies in general do not show any directional preference in exerting their gravitational influence.

The law of gravitation applies with equal accuracy to bodies which are separated by astronomical distances as well as to those which are very close to each other.

It is reported that the constant of gravitation varies slightly with temperature.

VARIATION OF '*g*'

115. Ritche in 1672 observed that the period of his seconds pendulum changed when it was taken from Cayenene to Paris. Later,

experiments clearly established that the value of g changes whenever there is a change of (a) latitude or (b) altitude.

116. Variation with latitude

The value of 'g' changes with latitude due to two causes: (1) rotation of the earth and (2) bulging at the equator.

It is believed that the earth was at one time a liquid mass, the outer layer of which first solidified as it cooled. Such a liquid mass would assume a spherical shape under the action of the mutual gravitational forces of its particles if it had been at rest. But actually the earth is rotating and every particle which is not on the axis of rotation experiences a centrifugal force equal to $m\omega^2 r$, where m is the mass of the particle, ω the angular velocity of rotation and r the distance of the particle from the axis of rotation. At a point P (Fig. 77) on the surface of the liquid mass in latitude λ, the distance $r = a \cos \lambda$ (where a is the radius of the spherical mass).

$\therefore$ The centrifugal force on unit mass $= \omega^2 a \cos \lambda$.

Now there would be two forces acting on the particle at P: (1) the attraction of the whole mass of the earth on the particle and (2) the centrifugal force (which is comparatively much smaller than that due to the first. The resultant of these two forces should be perpendicular to the surface at P as otherwise the liquid mass would not be in equilibrium. The surface of the liquid earth would therefore deviate from the spherical form and assume the form of an oblate spheroid, on the surface of which the resultant attraction at any point would be perpendicular to the surface at that point. The earth, when it cooled, solidified in the same form. This is how the bulging of the earth at the equator and the flattening at the poles (i.e., the ellipticity of the earth) is accounted for. The equatorial radius of the earth is 6,378,300 metres and the polar radius 6,356,500 metres.

117. Variation of 'g' due to the rotation of the earth

Now, the centrifugal force on a unit mass at P $= \omega^2 r$ and it acts along PM.

The unit mass is also attracted towards 'O' by a force equal to 'g'.

It is thus under the action of two forces :

1. $\omega^2 r$ along PM and
2. 'g' along PO (Fig. 77)

Let g_1 be the resultant force, acting on it. It will be equal to the observed value of the acceleration due to gravity.

From the law of parallelogram of forces, we have

$$PN^2 = PO^2 + PM^2 - 2PO \cdot PM \cos \lambda$$

(i.e.,) $g_1^2 = g^2 + \omega^4 r^2 - 2\omega^2 rg \cos \lambda$.

But ω is a very small quantity as compared to g, so that $\omega^4 r^2$ may be neglected.

$$\therefore \quad g_1^2 = g^2 - 2\omega^2 rg \cos \lambda$$

$$= g^2 \left(1 - \frac{2\omega^2 r}{g} \cos \lambda\right)$$

$$\text{or, } g_1 = g \left(1 - \frac{2\omega^2 r}{g} \cos \lambda\right)^{\frac{1}{2}}$$

$$= g \left(1 - \frac{\omega^2 r}{g} \cos \lambda\right) \text{ (approximately)}$$

$$= g - \omega^2 r \cos \lambda.$$

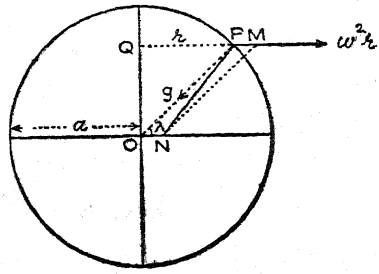


Fig. 77

Now $r = a \cos \lambda$ so that the apparent weight of a unit mass g_1 at latitude λ is given by the relation,

$$g_1 = g - \omega^2 a \cos^2 \lambda \quad \ldots \quad \ldots \quad (1)$$

acting along PN of the parallelogram PONM. Thus on account of the rotation of the earth, the observed value of g will be less than the gravitational force of attraction by an amount ($\omega^2 a \cos^2 \lambda$) on account of the centrifugal force produced by the rotation of the earth.

But *the apparent weight of a body at a place was found to be different from the value obtained from (1). This discrepancy is due to the ellipticity of the earth and the local variations in the density of the earth.*

Due to earth's ellipticity, the particles on the surface of the earth at the poles are nearer to the centre of the earth than the particles on the equator. The force of attraction on a body will be therefore least at the equator; it increases gradually towards the poles where it attains the maximum value.

Considering the two effects depending upon latitude—one due to rotation and the other due to change of distance from the centre of

the earth—Helmert suggested the following equation for the apparent weight of unit mass of a body at any latitude λ.

$$g_\lambda = g_e \ (1 + 0.005310 \sin^2 \lambda)$$

g_λ and g_e being the accelerations due to gravity at the place and at the equator respectively.

Problem : Calculate the centrifugal force experienced by a particle of unit mass placed at the equator using the following data :

Equatorial radius of the earth $= 6.378 \times 10^8$ *cm.*

Period of rotation of the earth = *1 Sidereal day.*

$= 86{,}164$ *sec.*

The centrifugal force experienced by a particle of unit mass on the equator $= \omega^2 r$ (since $\lambda = 0$ and $\cos \lambda = 1$) where ω is the angular velocity of rotation and r the radius of the earth.

Period of rotation of the earth, T = 86,164 sec.

Angular velocity, $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{86{,}164}$ radians/ sec.

Radius of the earth at the equator $r = 6.378 \times 10^8$ cm.

$\therefore$ Centrifugal force on unit mass $= \omega^2 r$

$$= \left(\frac{2\pi}{86{,}164} \right)^2 \times 6.338 \times 10^8 = 3.39 \text{ dynes.}$$

118. Variation with Altitude

Bouguer, in his mountain experiments, established for the first time that 'g' varies with altitude.

We have shown that $g = \dfrac{GE}{R^2}$,

(E being the mass of the earth and R its radius).

$$\therefore g \propto \frac{1}{R^2} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

If g_h be the acceleration due to gravity at an altitude, h above the ground $g_h \propto \dfrac{1}{(R + h)^2}$ $\ldots$ $\ldots$ (2)

$$\therefore \frac{g_h}{g} = \left(\frac{R}{R + h} \right)^2 = 1 - \frac{2h}{R} \text{ (approximately).}$$

But this relation gives values which are less than those obtained from experiments. This difference is due to the attraction of the

elevated masses of the earth such as mountains lying below the unit mass. Bouguer gave a relation taking this attraction into account.

If d is the density of the surface constituents, D the density of the earth and R its radius,

$$g_h = g\left(1 - \frac{2h}{R} + \frac{3hd}{2RD}\right).$$

This relation is known as **Bouguer's rule.**

119. Variation due to depth

If g_h is the acceleration due to gravity at a depth h below the surface of the earth,

$$g_h = G \cdot \frac{E}{(R-h)^2} = G \cdot \frac{4}{3}\pi (R - h) D.$$

$$\text{But } g = G \cdot \frac{E}{R^2} = G \cdot \frac{4}{3}\pi RD$$

$$\therefore \frac{g_h}{g} = \frac{R - h}{R} = 1 - \frac{h}{R}.$$

But Airy's experiments in Harton coal-pits indicated a slightly higher value for 'g' than at the surface. Experiments conducted by other scientists also gave a similar result. This discrepancy in the theoretical and experimental results has been suggested to be due to the much higher density of the central portion of the earth than that of the outer layers.

120. Gravity balance: Threlfall and Pollock's experiments

In 1809 Threlfall and Pollock designed a simple apparatus for the study of the variation of 'g' at different places.

The apparatus consists of a fine quartz fibre AB (30·5 cm. in length and 0·0038 cm. in diameter) stretched between two cylinders. A small spring s helps in keeping the fibre in tension. The cylinder at A is fixed, while that at B can be rotated by means of a rotating arm, R. A vernier V on the arm R moves over a circular scale and gives the position of the rotating arm on the scale.

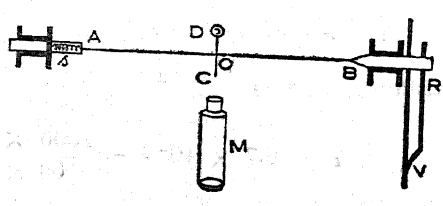


Fig. 78

A brass wire CD (5·3 cm. long) is fused to the quartz fibre at its centre O. It is weighted with gilded brass ball at D so that the centre of gravity of the wire is to a side of the quartz fibre.

The wire CD is brought into a horizontal position by twisting the fibre by means of the arm R. A microscope M is arranged to view the wire in this horizontal position. This position is, however, unstable and a slight movement of R from its position will make the wire CD turn over. A stop is therefore placed below CD to prevent this.

The weight of the ball at D increases or decreases as 'g' increases or decreases and the change of weight of the ball disturbs the equilibrium of the wire CD. To restore equilibrium, the quartz fibre is twisted by rotating the arm R through the required extent. The extent of rotation of the arm gives a measure of the change in the value of 'g'.

The instrument is calibrated by measuring the difference in the vernier readings at two places where the values of 'g' are known accurately. Threlfall and Pollock selected Sydney and Melbourne for this purpose.

The length of the wire CD and the rigidity of the quartz fibre are both affected by temperature. A correction for temperature will therefore have to be applied for obtaining accurate results.

Examples

1. *Two lead spheres of masses 2 kg. and 3 kg. are placed with their centres 100 cm. apart. Find the force of attraction between them if* $G = 6{\cdot}7 \times 10^{-8}$ *C. G. S. units.*

Force between two masses M_1 and M_2 separated by a distance, $d = G\dfrac{M_1 M_2}{d^2}$.

$$\therefore F = 6{\cdot}7 \times 10^{-8} = \frac{2000 \times 3000}{100 \times 100} = 4{\cdot}02 \times 10^{-5} \text{ dynes.}$$

2. *Calculate the mean density of the earth assuming it to be a uniform sphere of radius* $6{\cdot}4 \times 10^8$ *cm. given that a gm. on the surface of the earth is attracted with a force of 980 dynes and that the gravitational constant* $G = 6{\cdot}75 \times 10^{-8}$ *C. G. S. units.*

We have the relation, $g = G \cdot \frac{4}{3}\pi RD$.

Force on unit mass = acceleration due to gravity, g
$= 980$ dynes.

Gravitational constant, $G = 6.75 \times 10^{-8}$ C. G. S. units.

Radius of the earth, $R = 6.4 \times 10^{8}$ cm.

$$\therefore D = \frac{g}{G\,\frac{4}{3}\pi R} = \frac{980}{6.75 \times 10^{-8} \times \frac{4}{3}\pi \times 6.4 \times 10^{8}}$$
$$= 5.42 \text{ gm. / c.c.}$$

3. *Calculate the mass of the sun from the following data :*
Distance between the sun and the earth $= 1.489 \times 10^{13}$ *cm.*
Gravitational constant, $G = 6.67 \times 10^{-8}$ *C. G. S. units.*
Period of rotation of the earth round the sun $= 365$ *days.*

The force of attraction of the sun on the earth gives the necessary centripetal force for earth's revolution round the sun.

Let the mass of the sun $= M_1$
Let the mass of the earth $= M_2$
Angular velocity of rotation,

$$\omega = \frac{2\pi}{365 \times 24 \times 60 \times 60} \text{ radians / sec.}$$

Centripetal force

$$= M_2 \left(\frac{2\pi}{365 \times 24 \times 60 \times 60} \right)^2 \times 1.489 \times 10^{13} = 0.5905 M_2.$$

Force of attraction of the sun on the earth

$$= \frac{6.67 \times 10^{-8}\, M_1 M_2}{(1.489 \times 10^{13})^2} = 3.007 \times 10^{-34}\, M_1 M_2.$$

Hence $3.007 \times 10^{-34}\, M_1 M_2 = 0.5905\, M_2$

or $M_1 = 1.964 \times 10^{33}$ gm.

Exercises

1. State Kepler's laws of planetary motion.

2. (a) State the Universal law of Gravitation.

(b) Explain how Newton verified that the force on the moon due to the earth obeys the same law as the force on bodies at the surface of the earth.

3. (a) Define 'intensity' and 'potential at a point in a gravitational field'.

(b) Obtain an expression for the potential at a point due to a spherical shell at a point outside it.

4. (a) Define the 'Newtonian gravitation constant.'

(b) Derive an expression for the potential at a point within a solid sphere.

5. Obtain the relation between the gravitation constant and acceleration due to gravity, g. What are the dimensions of 'G'?

[*Hint:* $F = G\frac{m_1 m_2}{r_2}$

$\therefore [MLT^{-2}] = G\,[M^2L^{-2}]$

or dimensions of $G = [M^{-1}L^3T^{-2}]$]

6. (a) Explain how 'g' varies with latitude and altitude. Obtain a general relation for the variation of 'g' in each case.

(b) Describe how the variation of 'g' may be studied.

7. Two spherical masses (45 kg. and 300 kg.) are found to attract one another with a force of 0·359 dynes when their centres are 50 cm. apart. Find the gravitational constant

[*Ans.* $6{\cdot}65 \times 10^{-8}$ C. G. S. units]

8. Calculate 'g' at a place from the following data assuming that the earth is a perfect sphere :

$G = 6{\cdot}65 \times 10^{-8}$ C. G. S. units, $R = 6{\cdot}4 \times 10^8$ cm.

and $D = 5{\cdot}5$ gm. / c.c.

Also determine the potential due to the earth at a point on its surface.

[*Ans.* 980·6 cm. / sec.^{2}; and $6{\cdot}276 \times 10^{11}$ ergs. / gm.]

9. Describe Cavendish's experiment and explain how the Newtonian constant of gravitation is found from it. How is the mass of the earth estimated therefrom? [*A. U. March 1961*]

10. Find an expression for the gravitational force due to a thin hollow sphere of uniform density at a point outside it.

Sputnik I encircled the earth at a distance of 560 miles. Calculate its period of revolution. Radius of the earth = 4000 miles. (g = 32 ft. / sec^2) [*Ans.* 1·726 hrs.]

[*A. U. April 1962*]

11. Explain the term 'gravitation constant'. Describe Cavendish's experiment to determine this constant.

Show how the mass of the sun can be determined approximately in terms of the mass of the earth. [*A. U. April 1963*]

12. Describe the experiment of Boys on the determination of the gravitation constant. Deduce the relation between the constant of gravitation and acceleration due to gravity. How is the mean density of the earth estimated? [*A. U. Sept., 1963*]

13. Define 'constant of gravitation' and find its dimensions. Describe in detail, an accurate laboratory method of finding the constant of gravitation. [*A. U. April 1964*]

14. Enunciate Kepler's first and second laws of planetary motion and hence deduce Newton's law of gravitation. How does Kepler's law follows from this? [*M. U. April 1964*]

15. State Kepler's law of planetary motion. How are they related to Newton's law of universal gravitation?

Describe how the value of 'g' varies as we go below the surface of the earth, up to the centre of the earth. [*M. U. Sept., 1964*]

16. Discuss the variation of acceleration due to gravity with latitude and altitude. How are the values of acceleration due to gravity at two places accurately compared? [*M. U. Sept., 1964*]

17. State Newton's law of gravitation and explain what is meant by the gravitation constant. Find an expression for the acceleration due to gravity on the surface of a spherical planet of radius 'r' and of uniform density 'ρ'. Assuming that Mars is a sphere of radius 3,400 km. and density, 3·9 gm. cm^{-3} and that the earth is a sphere of radius 6,370 km. and density 5·5 gm. cm^{-3}, calculate the value of the acceleration due to gravity on the surface of Mars.

(Neglect the effect due to rotation of Mars and earth.)
(Acceleration due to gravity on the earth's surface = 980 cm. /sec^2.
(*Ans.* : 371·0 cm. / sec^2.) [*A. U. April 1965*]

18. Describe with theory, Cavendish's method of finding the gravitation constant. How does this help to find the mass of earth? Calculate the minimum velocity to be given to a body for it to escape

from the earth into space. (Radius of the earth = $6{\cdot}38 \times 10^8$ cm. Acceleration due to gravity = 981 cm. / sec^2.)

(*Ans :* $1{\cdot}119 \times 10^6$ cm. / sec.) [*M. U. April 1965*]

19. Describe the Boys method of determining the gravitational constant showing how it is deduced from the observation. Calculate the attraction between a pair of spheres of 15 kg. and 25 kg. whose centres are 20 cm. apart. (Assume G = $6{\cdot}7 \times 10^{-8}$ C. G. S. units)

(*Ans* : 6.281×10^{-2} dynes.) [*A. U. Sept., 1965*]

20. Define the 'constant of gravitation'. What are its dimensions?

Describe the method of Boys to determine the constant. In what respects is it an improvement on the method of Cavendish?

[*A. U. September 1965*]

21. Deduce Newton's law of Gravitation from Kepler's laws of planetary motion. Knowing that the moon at a distance of 2,40,000 miles makes one revolution round the earth in $27\frac{1}{3}$ days, show that an artificial satellite circling the earth at a height of 200 miles will complete a revolution in about 91 minutes.

(Radius of the earth = 4000 miles) [*M. U. April 1966*]

22. Write notes on

(a) Value of 'g' on the top of a mountain.

[*M. U. Sept., 1964*]

(b) Comparison of the values of 'g' at different places.

[*M. U. April 1965*]

(c) Poynting's method of finding the gravitation constant.

[*M. U. April 1964*]

(d) Mass of the earth. [*M. U. April 1966*]

(e) Variation of 'g' due to rotation of earth.

[*A. U. October 1967*]

23. Mention briefly how the acceleration due gravity varies over the earth's surface. [*A. U. March 1966*]

24. What is gravitation constant?

Describe a laboratory method of determinig the gravitation constant. How would a knowledge of the gravitation constant enable one to determine the average density of the earth?

[*A. U. October 1966*]

25. Describe Boy's method of determining the gravitation constant. Why is this method more accurate than that of Cavendish?

A satellite of Jupiter moves about in a circle of radius $1{\cdot}16 \times 10^6$ miles completing one revolution in 16·6 days. Calculate the mass of Jupiter. $G = 6{\cdot}66 \times 10^{-8}$ C. G. S. units. [*Ans.* $1{\cdot}874 \times 10^{30}$ gm.]
[*A. U. April 1967*]

26. Describe any one method of finding the gravitation constant. How does it compare with other methods in point of accuracy? If the pull of a sphere of mass 150 kg. on one mass of 20 kg. placed with their centres 30 cm. apart is equal to the weight of 0·22 mg. Calculate the mass of the earth, given its radius to be 6371 km.
[*Ans.* $6{\cdot}148 \times 10^{27}$ *gm.*] [*A. U. Oct., 1967*]

27. State Newton's law of gravitation. How is the average density of the earth estimated?

The moon describes a circular orbit of radius $3{\cdot}8 \times 10^5$ kilometres about the earth in 27 days and the earth describes a circular orbit of radius $1{\cdot}5 \times 10^8$ kilometres round the sun in 365 days. Determine the mass of the sun in terms of that of the earth. [*Ans.* $3{\cdot}366 \times 10^5$]
[*A. U. April 1968*]

28. Write a note on
'Variation of acceleration due to gravity.'
[*A. U. Oct., 1968*]

CHAPTER VI

KINETIC THEORY

121. The nature of matter was a subject for inquiry even in early times. Even though matter appears to be continuous, some thinkers like Thales of Miletus and Kanada of India were able to postulate that matter is discontinuous and that it consists of discrete particles. It is now firmly established that matter consists of molecules and atoms. Gases are more simple in their behaviour and lend themselves more easily to theoretical study than solids and liquids.

Maxwell and Clausslus of the nineteenth century directed their attention to them and developed the kinetic theory of gases. They not only explained many of the physical properties of gases successfully but also predicted results which were later verified by experiments.

That a gas cannot be a continuous medium but should consist of discrete particles is clear from several observations connected with gases. Gases mix with one another freely. They cannot occupy the same space, if they be continuous. A gas expands indefinitely when its pressure is reduced. It cannot obviously do so unless it consists of particles which are in motion. If it is then assumed that a gas consists of discrete particles in motion, it should be possible to explain, on this assumption, the properties of gases such as those given by the gas laws and the phenomena like diffusion, viscosity, conduction of electricity through gases, Brownian motion etc., by the kinetic theory.

122. Fundamental assumptions of the Kinetic Theory

(a) Every gas consists of a large number of discrete particles called **molecules** *which are exactly alike in respect of their size, shape and mass. The theory considers them to be small and perfectly elastic spheres.* Though later researches have shown that the structure of molecules and atoms is very different from this simple mechanical picture, the results derived on this assumption of this theory have been surprisingly successful.

124. Mathematical investigation

A few **simplifying assumptions** are necessary to make an elementary mathematical investigation of the motion of the molecules and develop the theory. They are as follows :

1. The molecules of a gas are identical hard spheres which are perfectly elastic so that there is no loss of kinetic energy at an encounter with the wall or between themselves.

2. The molecules travel in straight lines between successive encounters. The *'free path'* of a molecule is straight.

3. The time spent at an encounter on the average is negligible in comparison with the time taken for describing the free path.

4. The size of a molecule is negligible compared to the volume of the containing vessel.

5. Mutual forces of attraction (i.e., cohesive forces) between molecules are negligible in a gas.

The **mean free path** is an important concept introduced by Claussius. It has been defined as *the average distance travelled by a molecule between two successive encounters.*

125. Pressure of a gas

Suppose a gas is enclosed in a cubical vessel of side 1 cm. and let it contain n molecules.

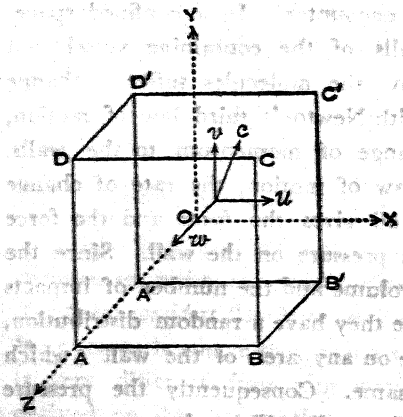


Fig. 79

Let m be the mass of each molecule.

Let a molecule travel with a velocity c in any direction and let u, v and w be its components, along the X, Y and Z axes respectively which are parallel to the sides of the cube so that $c^2 = u^2 + v^2 + w^2$.

When the molecule strikes a wall, perpendicular to the X—axis the velocity components v and w are not affected by the

impact but the u-component is reversed. The velocity in the direction of the X—axis becomes $-u$ after the impact.

Now the momentum of the molecule in the X-direction before impact $= mu$.

The momentum in the same direction after impact $= -mu$.

$\therefore$ Change of momentum suffered by the molecule in the X-direction at the impact $= mu - (-mu) = 2mu$. ... (1)

Since this component is not affected by impacts at the walls perpendicular to the Y and Z axes, *we may assume that the molecule moves to and fro between the set of parallel walls perpendicular to the X—axis so far as the impacts on those walls are concerned.*

The molecule will have therefore to travel 2 cm. parallel to the X—axis between the successive collisions on the wall considered.

$\therefore$ Number of collisions of the molecule against the wall in 1 sec. $= \frac{u}{2}$ ... (2)

$\therefore$ Change of momentum of the molecule in 1 sec. at the wall $= 2mu \times \frac{u}{2} = mu^2$,

If $c_1, c_2, c_3 \ldots c_n$ be velocities of the n molecules, and

$u_1, v_1, w_1 ; u_2, v_2, w_2 ; \ldots u_n, v_n, w_n$

be their velocity components in the three directions respectively, the total change of momentum suffered by all the n molecules due to encounters at this wall in 1 sec.

$= mu_1^2 + mu_2^2 + \ldots + mu^2{}_n = m \sum u_1^2$ .... (3)

If $\overline{u^2}$ be the average of the squares of these velocity components

$$\overline{u^2} = \frac{u_1^2 + u_2^2 + \ldots + u_n^2}{n}$$

or $\sum u_1^2 = n\,\overline{u^2}$ .... (4)

$\therefore$ The change of momentum in 1 sec. at the wall $= mn\,\overline{u^2}$ ... (5)

From Newton's third law, this will be equal to the change of momentum experienced by the wall also in 1 sec. (in the opposite direction) and hence the rate of change of momentum or force experienced by the wall is equal to $mn\,\overline{u^2}$.

Since the wall is of unit area, this will be equal to the pressure exerted on the wall.

∴ The pressure exerted on the wall perpendicular to the X—axis $= mn\,\overline{u^2}$.

An equal pressure will also be exerted on the opposite wall.

Similarly, the pressure on the walls perpendicular to the Y—axis $= mn\,\overline{v^2}$ and on the walls perpendicular to the Z—axis $= mn\,\overline{w^2}$ where $\overline{v^2}$ and $\overline{w^2}$ are the averages of the squares of the velocity components in the two directions respectively.

Since the number of molecules in the unit cube is very large, and since they have a random distribution of velocities, there is no reason for the average values in the three different directions to be different, so that it may be assumed that $\overline{u^2} = \overline{v^2} = \overline{w^2}$ ... (6)

$\therefore\ mn\,\overline{u^2} = mn\,\overline{v^2} = mn\,\overline{w^2} = p$ (say) ... (7)

i.e., the same pressure is exerted on all the walls.

Now from the relations, $c_1^2 = u_1^2 + v_1^2 + w_1^2$

$c_2^2 = u_2^2 + v_2^2 + w_2^2$

.....

.....

$c_n^2 = u_n^2 + v_n^2 + w_n^2$

it follows that $\sum c_1^2 = \sum u_1^2 + \sum v_1^2 + \sum w_1^2$ ... (8)

If $\overline{c^2}$ be equal to $\frac{\sum c_1^2}{n}$ (the average of the squares of the molecular velocities) $\sum c_1^2 = n\,\overline{c^2}$ ... ... (9)

and $n\,\overline{c^2} = n\,\overline{u^2} + n\,\overline{v^2} + n\,\overline{w^2}$ ... from (8)

i.e., $\overline{c^2} = 3\,\overline{u^2} = 3\,\overline{v^2} = 3\,\overline{w^2}$ ... from (6)

or $\overline{u^2} = \overline{v^2} = \overline{w^2} = \frac{\overline{c^2}}{3}$.

Thus according to the kinetic theory, the pressure exerted on any one of the walls of the unit cube $= mn\,\frac{\overline{c^2}}{3}$

or $p = \frac{1}{3}\,mn\,\overline{c^2}$ ... ... (10)

Since m is the mass of a molecule and n the number of molecules in 1 c.c., $m \times n$ = the mass of the gas per c.c.
$= \rho$, the density of the gas

$$\therefore \rho = \frac{1}{3} \rho \overline{c^2} \quad \ldots \quad \ldots \quad (11)$$

126. Root mean square velocity

$\sqrt{\overline{c^2}}$ is called the root mean square velocity.

Equation (11) for the pressure of a gas enables us to calculate the root mean square velocity of the molecules of the gas if the pressure and the density of the gas are known.

$$\text{Thus } \overline{c^2} = \frac{3p}{\rho} \quad \text{or} \quad \sqrt{\overline{c^2}} = \sqrt{\frac{3p}{\rho}} \quad \ldots \quad (12)$$

In the case of hydrogen, for instance, the density at N. T. P. = 0·00009 gm. per c.c.

$$\therefore \sqrt{\overline{c^2}} = \sqrt{\frac{3 \times 76 \times 13{\cdot}6 \times 980}{0{\cdot}00009}}$$
$$= 1{\cdot}84 \times 10^5 \text{ cm. per sec.}$$

It may be noted that $\overline{c}$, the mean molecular velocity is not the same as the root mean square velocity, $\sqrt{\overline{c^2}}$

$$\text{since } \overline{c} = \frac{c_1 + c_2 + \ldots +}{n}$$

$$\text{and } \sqrt{\overline{c^2}} = \sqrt{\frac{c_1^2 + c_2^2 + \ldots + c^2{}_n}{n}}.$$

However, according to Claussius and Maxwell $\overline{c} \propto \sqrt{\overline{c^2}}$

127. Deduction of gas laws

(a) **Boyle's Law**: The pressure exerted by a gas, $p = \frac{1}{3} \rho \overline{c^2}$, ρ being the density of the gas and $\overline{c^2}$ the mean square velocity of the molecules.

If v is the volume per unit mass of the gas, $\rho = \frac{1}{v}$.

$$\therefore p = \frac{1}{3} \cdot \frac{1}{v} \cdot \overline{c^2}$$

$$\text{or} \quad pv = \frac{1}{3} \overline{c^2} \quad \ldots \quad \ldots \quad (13)$$

Now according to the theory, the average kinetic energy of the molecules $\left(\frac{1}{2} m \overline{c^2} \right)$ remains constant when the temperature is constant.

$\therefore \overline{c^2}$ is constant when the temperature is constant. It thus follows that pv is a constant for a mass of a gas when the temperature is constant.

This is Boyle's law.

(b) **Charles' Law :** Again since $\frac{1}{2} m \overline{c^2}$ varies as the absolute temperature, T and m is a constant, it follows from (13) that

$$pv \propto T \quad \ldots \quad \ldots \quad (14)$$

i.e., the volume of a mass of gas varies as its absolute temperature when the pressure of the gas is constant and the pressure of a mass of gas varies as its absolute temperature when the volume is constant.

This is Charles' law.

128. Gas equation

Introducing the constant of proportion R in (14) we get

$$pv = RT \quad \ldots \quad \ldots \quad (15)$$

p, v and T being the pressure, volume of unit mass and absolute temperature of the gas respectively. This is the gas equation.

129. Temperature and Energy

Applying relation (13) to a gram-molecule of the gas, we have, if V be the volume occupied by the gram-molecule,

$$pV = \tfrac{1}{3} \rho \overline{c^2} \times V$$

$$= \tfrac{1}{3} M \overline{c^2}$$

$$= R_0 T \text{ (say)},$$

where M is the molecular weight of the gas and R_0 the gas constant for a gram-molecule.

$$\therefore R_0 T = \tfrac{1}{3} M \overline{c^2}$$

But M = Nm where N is the number of molecules in a gram-molecule of the gas (i.e., Avogadro's number) and m is the mass of each molecule.

$$\therefore R_0 T = \tfrac{1}{3} N m \overline{c^2}$$

$$\text{or } \frac{R_0}{N} \cdot T = \frac{1}{3} m \overline{c^2}$$

Now, let $\frac{R_0}{N} = k$ (Boltzman's constant)

then $\frac{1}{3} m \overline{c^2} = kT$,

or $\frac{1}{2} m\overline{c^2} = \frac{3}{2} kT$. or, the energy of a molecule, $E = \frac{3}{2} kT$.

130. Avogadro's Law

Claussius and Maxwell showed that when two gases are in equilibrium, the mean kinetic energy of the molecules of one gas will be equal to the mean kinetic energy of the molecules of the other gas. But when there is equilibrium between the two gases, their temperatures should be the same. Therefore the mean kinetic energy of one gas is equal to the mean kinetic energy of the molecules of the other gas when their temperatures are the same.

Consider now unit volumes of two gases at the same temperature and pressure.

Let $p_1 = \frac{1}{3} m_1 n_1 \overline{c_1^2}$ for the first gas,

and $p_2 = \frac{1}{3} m_2 n_2 \overline{c_2^2}$ for the second gas.

Since $p_1 = p_2$, $\quad m_1 n_1 \overline{c_1^2} = m_2 n_2 \overline{c_2^2}$ ... (1)

At the same temperature, the mean kinetic energies of molecules in the two gases are the same.

$$\therefore \tfrac{1}{2} m_1 \overline{c_1^2} = \tfrac{1}{2} m_2 \overline{c_2^2}$$

$$\text{or } m_1 \overline{c_1^2} = m_2 \overline{c_2^2} \quad \ldots (2)$$

$\therefore$ From (1) and (2), $n_1 = n_2$.

This relation can be extended to any number of gases and thus *it follows that equal volumes of all gases at the same pressure and temperature contain the same number of molecules.* This is Avogadro's law.

131. Dalton's Law

The pressure of a mixture of gases and vapours which do not chemically interact with one another is equal to the sum of the

pressures which each one of them would exert if it alone occupied the same space at the same temperature.

Let m_1, m_2....be the masses,

n_1, n_2 ....the numbers of molecules per unit volume,

$\overline{c_1}^2, \overline{c_2}^2$....the mean square velocities of the molecules occupying the same volume at a certain temperature.

The pressure exerted by them when calculated from the fundamental assumptions of the kinetic theory will be

$$p = \tfrac{1}{3} m_1 n_1 \overline{c_1}^2 + \tfrac{1}{3} m_2 n_2 \overline{c_2}^2 + \ldots \quad \ldots \quad (1)$$

But if the individual gases alone are present and the pressures exerted by them are p_1, p_2....respectively,

$$\left.\begin{aligned} p_1 &= \tfrac{1}{3} m_1 n_1 \overline{c_1}^2 \\ p_2 &= \tfrac{1}{3} m_2 n_2 \overline{c_2}^2 \end{aligned}\right\} \quad \ldots \quad (2)$$

From (1) and (2) it is seen that $p = p_1 + p_2 + \ldots$

This is Dalton's law.

132. Mean free path of a molecule

The mean free path of a molecule has been defined as the average distance a molecule travels between two successive collisions. Its value may be deduced in a simple way as follows:

Let σ be the diameter of a molecule. As a molecule, say A, moves forward, it can have an encounter with another molecule only when it approaches the latter so near that the distance between their centres is equal to or less than σ.

Let us suppose that A is moving with a velocity c, equal to the average velocity of the molecules.

Imagine that, in the gas, the molecule A alone is moving and that all the others are at rest. As the molecule moves a distance c in 1 sec., we suppose that a cylinder of length c is described which has for its axis the path of the centre of the molecule A, and for its radius of cross-section a value equal to σ. If the centre of another molecule lies within this cylinder, the condition

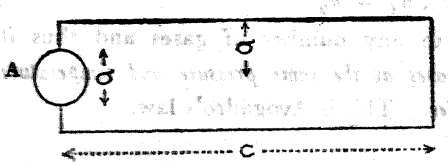


Fig. 80

for collision will be satisfied and the molecule A will collide with it. Thus there will be as many collisions for the molecule A in 1 sec., as there are molecules contained in this cylinder. Now the volume of the cylinder $= \pi\sigma^2 c$.

If n be the number of molecules per c.c. in the gas, the number contained within the cylinder $= \pi\sigma^2 cn$

$\therefore$ No. of collisions per sec. $= \pi\sigma^2 cn$

and the average time between two successive collisions $= \dfrac{1}{\pi\sigma^2 cn}$.

Hence the mean free path (or the average distance between two successive collisions),

$$\lambda = \frac{1}{\pi\sigma^2 cn} \times c = \frac{1}{\pi\sigma^2 n}.$$

It may be noted that in obtaining the expression, it has been assumed that only a single molecule is in motion while all the other molecules remain at rest. Taking the motions of all the molecules into consideration, Claussius and Maxwell obtained different expressions for λ proceeding in slightly different ways. However, in all these cases it is definitely shown that $\lambda \propto \dfrac{1}{\sigma^2 n}$. Since σ is a constant for the molecules of a gas, it follows that $\lambda \propto \dfrac{1}{n}$.

But n (the number of molecules per c.c.) depends upon the pressure of the gas, p.

$\therefore$ The mean free path, $\lambda \propto \dfrac{1}{p}$.

i.e., *the mean free path of a molecule varies inversely as the pressure of the gas, temperature remaining constant.*

133. Brownian Movement

The phenomenon of Brownian movement gives direct evidence to the kinetic theory of matter.

In 1827, Brown, a British Botanist, observed through a microscope that small pollen grains (0·0001 cm. in dia.) held in suspension in water, were in constant irregular motion. This erratic movement

of the particles was attributed by Brown to the vital forces that the grains possessed.

Subsequent work has revealed the nature of the phenomenon. The same kind of incessant irrigular motion was observed in the case of all small particles held in suspension in water. Such suspensions are called colloidal solutions.

The unceasing irregular motion has been reconsidered to be due to the molecular agitation that is going on within the liquid.

The molecules of water are so small that they cannot be seen by even the most powerful microscope invented till now. But the suspended particles, while they are big enough to be seen by a microscope, are also sufficiently small to respond to the collisions of the surrounding water molecules. Since the velocities of the latter are distributed at random, the resultant change of momentum due to their impacts on a suspended particle will not in general be zero and consequently the suspended particle will move in the direction of the net momentum gained by the particle at that instant. The process of impacts goes on without interruption and the suspended particle goes on moving and continually changes its direction of motion. An unceasing zig-zag movement of the particles is thus seen under the microscope.

Brownian movement was also found to be exhibited by suspended particles in gases. The movement of the particles in gases was found to be more vigorous than those in liquids which is in accordance with the kinetic theory.

Perrin conducted experiments to determine Avogadro's number (N) based on Brownian movement and obtained values which were in fair agreement with values obtained by other methods.

Brownian movement is therefore taken as a direct evidence of the incessant and haphazard motion of the molecules in liquides and gases.

DEVIATIONS FROM BOYLE'S LAW

134. According to the kinetic theory of gases, the product of pressure and volume of a gas should remain constant at constant temperature. This relation which is called Boyle's law should hold good for all gases. It has been actually found, however, that while this relation holds good when gases are studied at very low pressure, *there are considerable deviations from the law when they are studied at high or even moderate pressure. A gas that obeys Boyle's law is called an ideal gas.* The behaviour of gases at high pressures has been the subject of study by several scientists.

135. Experiments

(a) Despretz in 1827 conducted a series of experiments on different gases and compared their behaviour with that of air.

He arranged four barometers side by side in the same cistern and admitted equal volumes of different gases into different tubes. The mercury levels in all the tubes were at the same height to start with. They were then put in a strong glass vessel filled with water, fitted with a screw so that all the gases were subjected to the same increase in pressure. The gases were compressed but to different extents. Certain gases, cyanogen and ammonia underwent a great reduction in volume and were even liquefied. He observed that air and hydrogen behaved in the same manner up to 15 atmospheres but above that pressure hydrozen was less compressible than air.

These experiments clearly show that Boyle's law is not satisfied by any gas whatsoever.

(b) **Regnault's experiment**

Regnault, who conducted a series of experiments on a number of gases, erected two barometer tubes A and B in a closed vessel V into which mercury could be forced from a side tube at M. One of the tubes A was provided with a tap S at the upper end; it was filled with the experimental gas which was admitted into it from time to time when required. The other tube B was a long manometer tube which served to measure pressure.

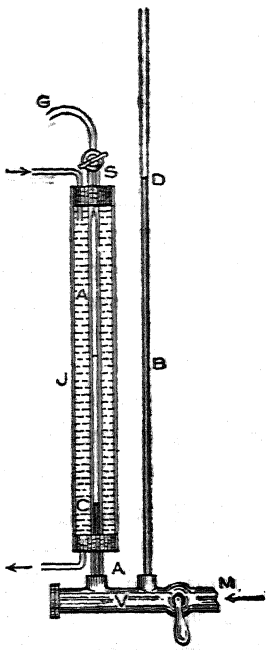


Fig. 81

To begin with, a certain amount of the gas having a volume v was admitted into A so that the mercury level in it stood at a mark C. The level of mercury D in the manometer tube B was noted and the pressure of the gas (= CD + atmospheric pressure, H) was obtained. *Mercury was then forced into the vessel and the pressure increased so that the gas was compressed to half of its original volume.* The pressure of the gas was obtained from the level of mercury in B as before.

Gas was now admitted into A so that the level of mercury in it was forced down to C and the pressure of the gas noted. The volume of the gas at the beginning of the second trial had the same value as in the first one. Mercury was again forced until the gas was again compressed to half the volume. In this manner, the pressure was raised in a series of trials to a maximum pressure of 30 atmospheres and from each trial the product Pv was calculated. The product should be equal if it behaved as an ideal gas.

Regnault conducted experiments with different gases and at different temperatures also between 2°C and 10°C. The product Pv for all gases was found to vary and never remained constant as required by Boyle's law. He conducted experiments with air, nitrogen, carbon-dioxide etc., with pressure ranging from 1 to 30 atmospheres and found that the product Pv decreases as the pressure was increased in the case of all gases except hydrogen. For hydrogen alone, Pv increased with increase of pressure.

(c) **Natterer,** while conducting experiments to liquefy permanent gases, used pressures higher than those employed by Regnault. He found that the product Pv for nitrogen and oxygen reached a minimum at about 50 atmospheres of pressure and then increased.

Their behaviour above that pressure was found similar to that of hydrogen.

136. Amagat's experiments

Amagat employed pressure much higher than those employed by Regnault and Natterer. He calibrated a nitrogen gauge in a preliminary experiment for the purpose of measuring high pressures.

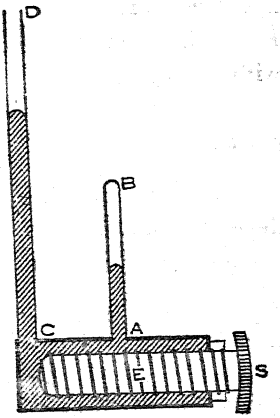
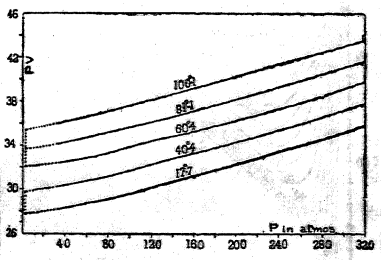


Fig. 82

To construct the gauge, he built a steel tube CD about a thousand feet long in the vertical shaft of a mine. The lower end C of the tube was introduced into a hard steel chamber E at the bottom of the mine. The tube AB closed at the upper end B, which contained nitrogen, was connected to E, close and parallel to CD. The chamber E was filled with mercury and by means of the screw S, the pressure inside was increased to any desired value.

The height of the mercury column above the level of mercury in CD together with atmospheric pressure gave the value of the pressure to which nitrogen gas was subjected. The level of mercury in CD was noted for each value of the pressure, so that the values of pressures could be read off directly from the level of mercury in CD.

The nitrogen tube so calibrated was then used to read the pressure in subsequent experiments. Pressures up to 400 atmos were applied by Amagat. In experiments with other gases the tube containing the gas was fitted to the chamber E, by the side of the nitrogen gauge and pressures were applied by turning the screw S.



PV—P diagram for hydrogen
Fig. 83

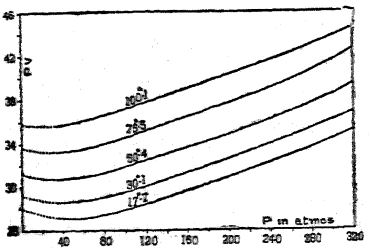
Amagat studied the behaviour of hydrogen, oxygen, carbon-dioxide etc. The curves obtained by him for different gases and at different temperatures are as in Fig. 83, 84 and 85.

The conclusions of his experiments may be summed up as follows :

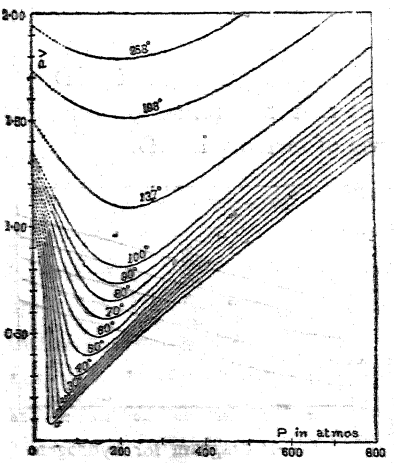
1. At low pressures, the product PV for all gases (except hydrogen) diminishes as P increases and beyond a certain value of P which depends upon the temperature, the product PV increases with pressure.

2. All gases except hydrogen have therefore a minimum value for PV.

3. The pressure corresponding to this minimum value for PV depends upon the nature of the gas and its temperature.



PV—P diagram for oxygen
Fig. 84



PV–P diagram for CO_2
Fig. 85

4. Hydrogen alone, at the temperatures studied, shows an increase in the value of PV as P increases even from the beginning. It cannot be said on this account that hydrogen belongs to a different category of gases. It can be noted from the curves that as the temperature at which a gas is studied is raised, the minimum point shifts towards the Y-axis and it may therefore be concluded that even in the cases of oxygen and nitrogen if the temperature is sufficiently high, the minimum point would disappear and the curve would show a rise from the beginning.

The behaviour of gases in this respect depends upon the temperature at which they are studied as compared to their **critical temperature.** (Critical temperature is the temperature below which a gas can be liquefied by increasing the pressure. A gas cannot be liquefied above the critical temperature however great the pressure applied may be.) A gas which is very remote from the critical temperature shows a behaviour

similar to that of hydrogen. (Fig. 83) A gas close to its critical temperature will give curves with steep drop as in the case of carbon-dioxide. (Fig. 85) Gases studied at intermediate temperature will give curves similar to those obtained with oxygen and nitrogen. (Fig. 84)

Van der Waals' Equation

137. The elementary kinetic theory developed in Art. 124 has been successful in accounting for Boyle's law and Charles' law and leading to the gas equation $PV = RT$. The experiments described above show, however, that *no gas actually obeys Boyle's law except at low pressures.* Several attempts were made to account for the actual behaviour of gases theoretically and modify the gas equation but no theory developed so far has been a complete success. Of these attempts the theory of Van der Waals is the one which is most satisfactory. He considered that the elementary theory is defective because it did not take into consideration (1) *the mutual forces of attraction between the molecules and* (2) *the finite size of the molecules.*

(a) A molecule in the interior of a gas is on the average surrounded equally on all sides by other molecules and the forces of attraction (cohesive forces) which the latter exert on it will consequently balance one another. The motion of the molecule is therefore not affected by the cohesive forces when it moves in the interior. When the molecule however approaches a wall, there are no molecules of its own kind present on the side of the wall to exert forces on it towards the wall. The forces of attraction exerted on it by the molecules lying towards the interior are therefore not compensated and the molecule experiences a resultant force towards the interior. Consequently the molecule will be retarded on its own journey towards the wall and the momentum with which it strikes that wall actually will be less than what it would have been in an ideal gas, i.e., a gas in which there are no cohesive forces.

The rate of change of momentum and therefore the pressure actually exerted would be less than in the case of an ideal gas. If the observed pressure be P, the pressure exerted by an ideal gas may be taken to be $P + p$, where p is the correction to be applied for the cohesive forces.

Van der Waals assumed that p is proportional to (1) the number of molecules attracted and (2) the number of molecules attracting. Now each one of these numbers is proportional to the number per unit volume, i.e., to the density (ρ) of the gas.

$$\therefore p \propto \rho \times \rho \quad \text{or} \quad p \propto \rho^2$$

But the density ρ varies inversely as the specific volume v of the gas.

$$\therefore p \propto \frac{1}{v^2} \quad \text{or} \quad p = \frac{a}{v^2}$$

where a is a constant which Van der Waals assumed to be independent of temperature.

Hence the pressure that would be exerted by an ideal gas would be $\left(P + \frac{a}{v^2} \right)$.

(b) Molecules have a definite size, however small they may be, and they occupy a certain fraction of the volume occupied by the gas. The volume in which the molecules are therefore free to move is less than the observed volume, v of the gas, by an amount which depends upon the total bulk of the molecules in the gas. If this correction be b, the volume corrected for finite size of the molecule becomes $(v-b)$.

Therefore, instead of $Pv = RT$, the gas equation according to Van der Waals becomes,

$$\left(P + \frac{a}{v^2} \right) (v-b) = RT$$

where a and b are constants for the gas and are to be determined by experiment.

The equation is found to express the behaviour of gases much better than the gas equation, but is not in complete agreement with the behaviour of all gases.

It should therefore be considered to be not quite correct.

Exercises

1. Derive Van der Waals' equation of state of a gas.

2. Find the number of molecules per c.c. in a gas given that the mean free path of the gas is $1{\cdot}83 \times 10^{-5}$ cm. and that

its molecular diameter is 2.3×10^{-8} cm. using Maxwell's expression for $\lambda \left(= \frac{1}{\sqrt{2}\pi\sigma^2 n} \right)$. [*Ans.* 2.325×10^{19}]

3. Calculate the molecular K. E. of hydrogen at N. T. P. ($R = 8.3 \times 10^7$ ergs.) [*Ans.* 33.99×10^9 ergs.]

4. (a) Explain how real gases deviate from Boyle's law.

(b) Derive Van der Waals' equation for a gas and indicate graphically the P - V relation for the gas at different temperatures. [*A. U. March, 1961*]

5. Deduce an expression for pressure exerted by a gas from elementary kinetic theory. Explain why a gas is heated on being compressed and cooled on expansion, from kinetic theory. [*A. U. Sept., 1962*]

6. On the basis of the kinetic theory derive an expression for the pressure of an ideal gas. Hence deduce Boyle's law. Also obtain an expression for the mean free path for a molecule of a gas. [*A. U. Sept., 1964*]

7. How do you explain Boyle's and Charles' laws from the point of view of kinetic theory. Discuss the concept of absolute temperature on the basis of the kinetic theory. Calculate the density of Helium at 50 atmos. and 27° C given that the gas constant $R = 8.31 \times 10^7$ ergs. per degree. (1 atmos. $= 1.013 \times 10^6$ dynes/sq. cm.) (*Ans*: 0.008125 gm./c.c.) [*A. U. April 1965*]

8. From considerations of the kinetic theory, obtain an expression for the pressure of a gas. Calculate the total kinetic energy of the molecules contained in 1 gm. of oxygen at 0° C. ($R = 8.32$ Joules per degree) (*Ans*: 223 Joules) [*A. U. April 1965.*]

9. What are the assumptions of the kinetic theory. Derive an expression for the pressure of a gas in terms of density and molecular velocity.

A gas is heated so that, the density remaining constant, the pressure is doubled. What is the increase in the root mean square velocity of the molecules? (*Ans*: $\sqrt{2}$ times) [*A. U. Sept., 1965*]

10. (a) Describe Amagat's experiments on the deviation of gases from Boyle's law. What are his main conclusions?

(b) Describe an experiment to determine the compressibility of water. *] A. U. Sept., 1965]*

11. What are the assumptions underlying the kinetic theory of gases? How far are they true in the case of real gases? Derive an expression for the pressure exerted by a gas on the basis of the kinetic theory. The density of Nitrogen at N. T. P. is 1·25 gm./litre. Evaluate the root mean square velocity of its molecules.

(*Ans :* $4{\cdot}93 \times 10^4$ cm./sec.) *[A. U. March 1966]*

12. Describe Regnault's and Amagat's experiments to study gases under high pressures. Discuss the importance of the results obtained. *[A. U. Oct., 1966]*

13. What are the assumptions in the kinetic theory of an ideal gas?

Deduce the perfect gas equation according to this theory.

Calculate the R. M. S. velocity of Hydrogen molecules at 200°C. The density of hydrogen at N. T. P. = 0·0899 gm./litre.

(*Ans :* $2{\cdot}419 \times 10^5$ cm./sec.) *[A. U. April 1967]*

14. Write a note on Brownian movement.

15. On the basis of the kinetic theory of gases, derive an expression for the pressure exerted by a gas. Derive Avogadro's law.

Find the average energy per molecule of a gas at 0°C. Universal gas constant = $8{\cdot}32 \times 10^7$ ergs per degree per gm. molecule. Avogadro's number = $60{\cdot}9 \times 10^{22}$.

(*Ans :* $5{\cdot}595 \times 10^{-14}$ ergs.) *[A. U. Oct., 1967]*

Chapter VII

DIFFUSION AND OSMOSIS

138. When two substances, which have the property of mixing with each other in any proportion whatsoever, are kept in contact, they will inter-penetrate spontaneously until the mixture attains uniform composition throughout. This is due to the migratory movements of molecules and is known as **inter-diffusion** or **diffusion.**

The process, whereby the molecules of one substance migrate spontaneously from regions of higher concentration into regions of lower concentration, is known as diffusion.

Diffusion in gases is highly rapid while in liquids it is very slow. In solids, diffusion is negligibly small.

Consider for instance a gas jar, containing hydrogen which is inverted over another containing carbon dioxide, left undisturbed for sometime. Carbon dioxide rises up into the upper jar even against gravity while hydrogen penetrates into the lower one until finally the mixture becomes uniform.

Similarly, if two liquids, which can mix, are placed in contact in a vessel, the layer of one liquid lying over that of the other and left undisturbed, each liquid penetrates into the other until they are uniformly distributed in the vessel. To make this clearly visible, a coloured liquid like copper sulphate solution or potassium dichromate solution is placed at the bottom of a tall glass jar. Clean water is then poured over the solution **gently** and carefully so that the line of separation between the solution and water is clearly seen. If the jar is left undisturbed and observations are made from time to time, the characteristic colour of the solution will be found to have spread upwards. A gradual change in the depth of the tint will also be seen in the solution and this change continues until the entire solution in the jar acquires uniform colouration.

Solids are also found to diffuse into one another in negligible amounts. *Austen* found that when gold and lead were kept in

contact for a number of years, a little trace of each metal was found in the other.

Kinetic theory of matter explains readily the process of diffusion. In the case of gases, the molecules are widely separated and also their velocities are considerable so that one can easily penetrate into the interspaces between the molecules of the other.

In the case of liquids, the interspaces between molecules are not so wide and the molecular movements are also restricted due to intermolecular attraction. This explains the slow rate of diffusion in liquids.

There is practically no migratory movement in the case of solids which explains absence of diffusion in them.

A - DIFFUSION IN LIQUIDS

139. Graham's experiments

Graham in 1851, conducted a series of experiments on diffusion of liquids. He filled a wide-necked bottle, A, with a concentrated solution of a salt (say, copper sulphate) to be investigated. The bottle was covered with a glass plate and then slowly introduced into a larger vessel, B, containing water *taking care to see that the water did not get into the bottle nor the solution get out of the bottle.* When the liquids settled down to rest, the glass plate was slowly moved sideways and removed so that the solution and water were brought into cantact with each other. The vessel B was also closed with a glass plate C to minimise evaporation.

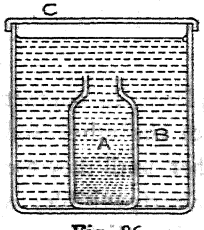


Fig. 86

Graham then took out samples of the solution from the top layers in the vessel B from time to time with a pipette and determined the quantity of salt in a known volume of the solution. He conducted experiments with a number of solutions and established several important facts. He was, however, unable to formulate any definite law for diffusion.

His conclusions are as follows :

1. The rate of diffusion of a salt is proportional to its concentration.

2. The rate of diffusion increases with temperature.

3. The rate of diffusion of a salt is independent of the presence of other salt solutions outside but is reduced if the same solution is present outside.

4. Solutions of different salts diffuse at different rates and may be partially separated if present together.

5. Crystalloids (i.e., substances like salts, sugar etc.) diffuse very much faster than colloids) i.e., substances such as albumen, glue and gelatine).

140. Fick's law of diffusion

Fick considered the various factors which govern the process of diffusion and found that they are analogous to the factors that govern the process of conduction of heat in a solid.

Fourier's equation of heat conduction is

$$Q = k.\,A.\,\frac{T_1 - T_2}{x}\,.\,t \qquad \ldots \qquad \ldots \qquad (1)$$

where Q is the quantity of heat conducted in a time t across two plane parallel faces, each of area A and separated by a distance x, k the coefficient of conduction of heat and T_1 and T_2 the temperatures of the faces.

Hence the rate at which heat is conducted across unit area is given by $Q_0 = k\,\frac{T_1 - T_2}{x}$ .... (2)

$\frac{T_1 - T_2}{x}$ is called the *temperature gradient.*

The relation (2) may also be written as $Q_0 = k.\,\frac{dT}{dx}$.

On the same lines, Fick stated the law of diffusion as follows :

$$Q = R \cdot A\,\frac{c_1 - c_2}{x}\,.\,t \qquad \ldots \qquad (3)$$

where Q is the quantity of salt that diffuses in a time t; c_1, c_2 are concentrations in gm. / c.c. of the salt at two layers separated by a

distance x; A, the area of the layer across which diffusion takes place and R is a constant called the diffusivity of the substance.

$\frac{c_1 - c_2}{x}$ is called the **concentration gradient.**

The same relation (3) may otherwise be written as

$$Q_0 = R \cdot \frac{dc}{dx} \quad \ldots \quad \ldots \quad (4)$$

where Q_0 refers to the quantity of salt that diffuses across unit area in unit time.

141. Diffusivity

The quantity of salt, that passes across unit area in unit time when the difference in concentrations between the two layers which are unit distance apart is unity is called the diffusivity of the substance.

Otherwise, *it is the quantity of salt that passes across unit area in unit time when the concentration gradient is one unit.*

The similarity of relations in heat conduction and diffusion shows that any result which holds for conduction of heat will also hold for diffusion.

142. Differential equation for diffusion

Consider two layers of unit area of the solution separated by a distance δx.

Let the concentration at the lower layer be c.

Let the concentration gradient be $\frac{dc}{dx}$.

Then, the decrease in concentration in the upper layer $= \frac{dc}{dx} \cdot \delta x$.

$\therefore$ Concentration at the second layer $= \left(c - \frac{dc}{dx} \delta x\right)$

The quantity of the salt that passes the lower layer per sec. across unit area $= R \cdot \frac{dc}{dx}$.

Similarly, the quanity of the salt that leaves the upper layer per sec. per unit area

$$= R \cdot \frac{d}{dx}\left(c - \frac{dc}{dx} \cdot \delta x\right) = R \cdot \frac{dc}{dx} - R \cdot \frac{d^2c}{dx^2} \delta x$$

Since this quantity is less than the quantity that enters from the lower layer, an amount of the salt

$$= R.\frac{dc}{dx} - \left(R.\frac{dc}{dx} - R.\frac{d^2c}{dx^2} . \delta x \right) = R.\frac{d^2c}{dx^2}. \delta x$$

accumulates in the solution between the two layers in one sec. and raises its concentration.

The volume of the solution contained between the two layers per unit area $= 1 \cdot \delta x = \delta x$.

If $\frac{dc}{dt}$ is the rate of rise of concentration, the quantity of the salt that has accumulated in this volume $= \frac{dc}{dt} . \delta x$.

$$\therefore \frac{dc}{dt} \delta x = R.\frac{d^2c}{dx^2} . \delta x \text{ or } \frac{dc}{dt} = R.\frac{d^2c}{dx^2}.$$

143. Determination of diffusion, R

The diffusivity R of any salt can be determined from the relation $Q_0 = R.\frac{dc}{dx}$. The experiment consists of two parts (a) determination of the rate of diffusion across a known area and (b) the concentration gradient $\left(\frac{dc}{dx} \right)$.

(a) A tall jar is filled with a concentrated solution of the salt in the lower half with a certain quantity of the salt at the bottom to keep the bottom layer always saturated. The upper half is filled with water (i.e., the solvent) without disturbing the lower half. A glass tube LMN with a wide opening at M and bent as in Fig. 87 is made to rest on the jar. A continuous stream of the solvent, water, is then caused to flow through LMN under a constant pressure. The stream of the solvent through the tube, carries with it the salt arriving at the opening M. After a time, steady conditions will be obtained when the amount of the salt arriving at M becomes constant.

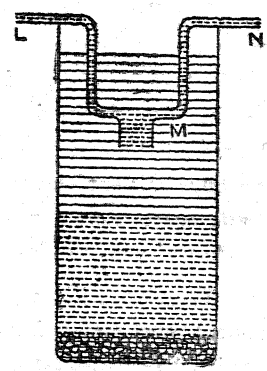


Fig. 87

The solution passing out at N is collected for a known time. By evaporating it, the mass of salt that has diffused, in that time, is found out. The area of the opening M gives the area of the layer across which diffusion into M has taken place. Thus the rate of diffusion across unit area, Q_0, is calculated.

(b) There are several methods for determining the concentration gradient $\frac{dc}{dx}$.

1. **Long's method** : He had provided two taps at a known distance apart in the upper half of the jar on one side. He collected the water coming out of the taps in a certain time and calculated the concentrations c_1 and c_2 of the layers of the solution at the same level as the taps. He was thus able to calculate the concentration gradient by measuring the distance x between the taps and calculating $\frac{c_1 - c_2}{x}$.

But this value was not accurate as the distribution of concentration in the column of the solution itself was disturbed when the solution was withdrawn from the taps. Further, the steady conditions anticipated in the experiment (a) were not obtained and the concentrations were found constantly changing.

The value of R was then calculated from the relation

$$Q_0 = R. \frac{c_1 - c_2}{x}.$$

2. **Kelvin's method** : Kelvin suggested that it would be preferable to make use of any property of matter which changes with concentration. Accordingly, he used a series of little beads of different densities as floats in the jar of the solution. The beads initially floated at the common surface of the solution in the lower half and the pure solvent above. But after a time, as diffusion progressed, the beads were found to arrange themselves at different layers where the densities of the layers coincided with their own densities.

Thus knowing the densities of the beads (by preliminary experiments) and measuring the distance apart in the jar, the concentration gradient is calculated. This method is also not very sensitive as bubbles of air collect round the beads and change their density.

144. Other methods

(i) *Weber*, in 1879, devised a method for determination of $\frac{dc}{dx}$ making use of the principle of the concentration cells. The electromotive force (e. m. f.) of a concentration cell depends upon the difference in concentration of the electrolyte in contact with electrodes in the cell.

Weber introduced two zinc plates horizontally at a known distance apart on one side of the upper half of the jar in experiment (a). He measured the e. m. f. between the plates from time to time and determined the corresponding difference of concentration between the layers.

(ii) In the later methods, optical properties such as refractive index and shift in the interference pattern which vary with concentration are also used. In the case of sugars, the concentration may be estimated by determining the angle through which the plane of polarisation of a beam of light is rotated by using a saccharimeter. This angle is proportional to the strength of the solution and its thickness.

B — OSMOSIS

144. (a) Semi-permeable membrane

It is a common experience that certain membrances allow certain liquids to pass through them, but refuse transmission to others. The following examples illustrate this fact :

(i) A bladder, filled with alcohol and introduced in water swells in size. But the same bladder, filled with water and immersed in alcohol, shrinks in size. The bladder in either case allows only water to flow through it but not alcohol.

(ii) A dried fruit soaked in water, if given sufficient time, swells to the full size, it had in the natural undried state. Water passes into the inside through the skin of the fruit.

(iii) Fingers of our hand dipped in a concentrated solution of soap (or salt) develop crinkles, as a consequence of the water inside the fingers flowing out through the skin of the fingers.

(iv) An egg, the shell of which is dissolved by dilute hydrochloric acid, swells in size if immersed in water but shrinks if immersed in a strong salt solution. In either case water flows through the outer membrane of the egg to the side where there is stronger solution.

Such membranes, as the bladder or the skins, are called **semi-permeable membranes.** They allow the solvent (water or liquid) to flow through them selectively but not the molecules of the solute dissolved in the solvent. Parchment paper, cellophane and de-nitrated, nitro-cellulose films also serve as semi-permeable membranes.

In all the examples cited above, water is the solvent and it can be seen that it flows out from regions of lower concentration to regions of higher concentration of the solute. *This process, by which solvents flow spontaneously from regions of lower concentration into regions of higher concentration, through a semi-permeable membrane, is called* **Osmosis.**

Osmosis plays an important part in the growth of plants and animal cells. The walls of the cells serve as semi-permeable membranes.

Graham separated the crystalloids and colloids in a mixture using a piece of bladder. This process, he termed, **dialysis.** The bladder transmits crystalloids and water in the mixture but not the colloids in it.

(b) **Osmotic Pressure**

A wide glass tube AB is closed at its lower end B with a semi-permeable membrane such as parchment paper. It is then filled with a strong solution of sugar (or salt) in water to a certain height, C and inserted in a beaker containing water. Osmosis sets in and the solvent begins to flow into the tube through the semi-permeable membrane. Consequently the level of the solution goes on rising. When the process is allowed to take place for some days, is found that the level in the tube AB becomes steady at a height say D. *The hydrostatic pressure due to the column of solution above the level of water outside, is said to be the osmotic pressure of the solute, sugar or salt.* This pressure is just sufficient to prevent further flow of the solvent into the tube.

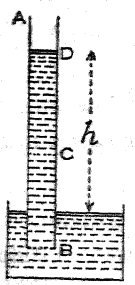


Fig. 87 (a)

The flow of the solvent through the semi-permeable membrane into the solution can be accounted for as follows :

Before osmosis commences, the pressure on solution side of the membrane is due to the pressure exerted by the molecules of water and the molecules of the solute ; whereas, the pressure on the water side is due to the molecules of the water alone. Due to the presence of the solute molecules, *the number of impacts of the water molecules per second on the solution side of the membrane is less than the number of impacts per second by water molecules on the solvent side.* More water molecules therefore enter the solution than those that leave it through the semi-permeable membrane. This process goes on until the number of water molecules entering the solution is equal to the number leaving the solution, i.e., the pressure on the two sides of the semi-permeable membrane due to water molecules is the same. The excess steady pressure on the solution side must be due to the molecules of the solute and therefore it is said to be the osmotic pressure of the solute.

(i) **Pfeffers method :** Pfeffer employed a freshly formed cupric ferro cyanide film as the semi-permeable membrane in his experiments. The film is mechanically weak and it cannot stand by itself. It is therefore caused to precipitate chemically in the pores of a porus-pot as follows.

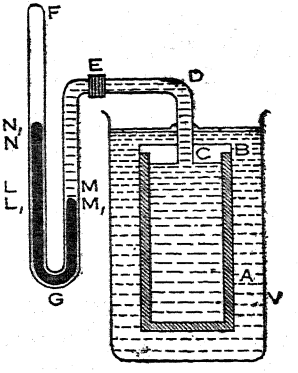


Fig. 87 (b)

A porus-pot filled with copper sulphate solution is immersed in potassium ferro cyanide solution. The two solutions begin to diffuse through the pores of the pot and the precipitate of cupric ferro cyanide is formed in the pores where the molecules of the salts come into contact.

The porus-pot, A, thus impregnated with a film of cupric ferro cyanide is filled with the solution to be investigated. It is next closed with a one-holed stopper B, fitted with a bent glass tube CDE. A closed U - tube manometer EGF (containing air over mercury in the closed tube) is connected at E for measuring

the pressure. The porus-pot is finally introduced into water contained in a vessel V as in the Fig. 87 (b). Osmosis commences and the levels of mercury in the manometer goes on changing until the steady state is reached, when the positions of the levels are, say, M and N. The osmotic pressure can be calculated as follows:

Let M and N be the levels of mercury in the manometer when the manometer is exposed to air at E and let L be a point vertically below N at the same level as M.

Now the pressure at M = Atmospheric Pressure, H.

Pressure of the enclosed air $p = H - NL$

Assuming the tube EF to be of uniform cross-section, the volume of the enclosed air can be taken to be proportional to the length occupied by it. The volume of the air in the tube in the beginning, $V \propto NF$.

Let $NF = l$

Let M_1 and N_1 be the altered final levels of mercury in the manometer; and let l_1 be a point below N_1 at the same level as M_1. Let the length of the air column, $N_1F = l_1$.

Now if p_1 be the pressure of the enclosed air, $p \times l = p_1 \times l_1$ (according to Boyle's law, temperature remaining the same)

$$\text{or} \quad p_1 = \frac{p \times l}{l_1}$$

Now the osmotic pressure $P = p_1 + N_1 l_1$

Pfeffer studied the osmotic pressures of a number of solutions in this way. The results of his experiments led to the formulation of the laws called the **laws of osmotic pressure,** which are given below. This method of measuring osmotic pressure is found to suffer from certain difficulties. The semi-permeable membrane is very weak and is found to burst when the osmotic pressures are high. Further, long intervals of time—days—have been found necessary for the levels of mercury in the manometer to become steady.

(ii) **Morse and Frazer's method:**

The porus-pots used in the experiments of Pfeffer were found to be lacking in uniform strength and porosity. Morse and his

co-workers, after long trials succeeded in preparing a mixture of clay for manufacturing the porus-pots. They adopted the electrical means to obtain the deposit of copper ferrocyanide membrane. The membranes, so obtained, are found to the mechanically stronger than the films used by Pfeffer. They used closed U-tube manometers with nitrogen sealed in a uniform calibrated tube for measuring high osmotic pressures. The porus-pot A in these experiments contained water and the metal cylinder C into which the pot was fitted contained the solution and the manometer, M. The copper ferrocyanide film, formed in the walls of the pot, was found to stand much higher osmotic pressures in the reverse direction than in Pfeffer's experiments. The osmotic pressure was measured in the same manner as in Pfeffer's experiments. The results obtained by Morse and Frazer were found to be closer in agreement to the gas laws, in the case of dilute solution than those of Pfeffer. However, the divergences in the case of stronger solutions were found to be considerable. As in the case of gases, the product PV is found to increase rapidly as the osmotic pressure increases.

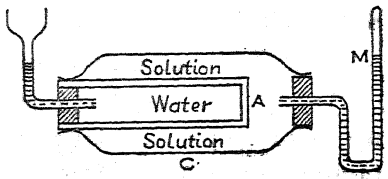


Fig. 87 (c)

(iii) Berkley and Hartley's method

Berkley and Hartley obtained more accurate results by measuring the pressures needed to be applied on the solutions to counteract osmosis.

The porus-pot with copper ferrocyanide films deposited in its walls contained water. It was surrounded by a jacket, fitted with a piston P and connected to a manometer M, containing this solution as in the Fig. 87 (d). The pressure on the solution was increased by moving the piston in, until the level of the water in the side-tube connected to the porus-pot remained the same as at the beginning. This shows that the water was not passing into the solution through the porus-pot. The pressure on the solution was read from the manometer. This pressure is equal to the osmotic pressure of the solution.

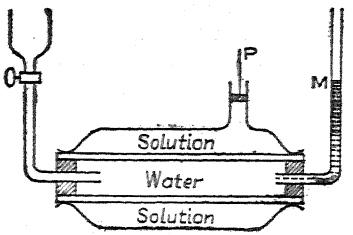


Fig. 87 (d)

The semi-permeable membrane in these experiments was not subjected to much strain as in the other experiments, as the osmotic pressure was balanced by the external pressure. The time taken in these experiments was also much less compared to the time required in the other experiments. Further, the concentration of the solution remained steady as there was no flow of water into it and the relations between osmotic pressure and concentration could be more accurately established.

(c) **Laws of osmotic pressure**

1. Pfeffer found from his experiments that the osmotic pressure for any solute is proportional to the concentrations of that solute in the solution. (provided the solution is not an electrolyte.)

(i.e., the osmotic pressure, $P \propto$ concentration, c) or $P \propto c$ ...(1)

2. Van't Hoff, making use of the results obtained by Pfeffer, showed that the osmotic pressure is proportional to the absolute temperature (i.e., the osmotic pressure, $P \propto$ absolute temperature, T)

or $P \propto T$ ...(2)

It follows from the above two laws that

$P \propto c.\ T$ or $P = k.\ c.\ T$ ...(3)

(where k is a constant).

If V is the volume of a solution containing 1 gram-molecule of the solute, $\frac{1}{V} = c$ (concentration being measured in gram-molecules per c.c. of the solution) and we get $PV = k.\ T$ ...(4)

Experiments showed that for solutions of low concentration (i.e., for infinite dilution), k is found to be identical with the gas constant, R.

The relation (4) may therefore be written as

$PV = RT$ which is the perfect gas equation.

It follows from this identity of relations that, *if the solute were present in the form of a gas at the same temperature and occupy the same*

volume, as that occupied by the solution, the pressure exerted by the gas will be the same as the osmotic pressure. This also leads to the formulations of **Avogadro's** law for dilute solutions. It may be stated that *in equal volumes of the solution at the same temperature, the osmotic pressure will be the same when equal number of molecules of the solute are present in equal volumes of the solvent.* It is thus seen that the osmotic pressure of a solute can be explained on kinetic theory just as pressure of a gas can be explained.

(d) Vapour pressure of a solution

The space above a liquid kept in an enclosure is saturated with the vapour of the liquid. At every instant, some liquid will be converted into vapour and simultaneously, some vapour will be converted back to liquid. There will thus be the process of evaporation and condensation taking place at the surface of the liquid continuously. They take place in such a way that there will be equilibrium between the two processes and the space above the liquid is saturated with the vapour of the liquid. The pressure exerted by the vapour in this state is maximum (at the prevailing temperature) and it increases with the increase in temperature.

It is observed that *the vapour pressure of a solution is less than that of the pure solvent at the same temperature.* This is theoretically explained as follows :

Let a tube A, closed at the lower end by means of a semi-permeable membrane, be dipped into the solvent contained in a vessel B, kept in a closed enclosure as in the Fig. 87 (e). Let the solution under investigation rise to the point M in the tube A, when the steady state is reached.

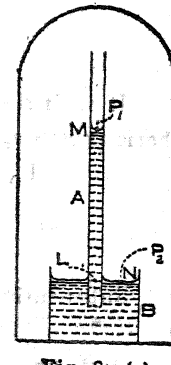


Fig. 87 (e)

The vapour at M is in contact with the surface of the solution in the tube A, while the vapour at N is in contact with the solvent in the vessel B.

Let P_1 and P_2 be the pressures of the vapour at M and N respectively.

Then $P_2 - P_1 = h\sigma g$

or pressure at N, $P_2 = P_1 + h\sigma g$ .... (1)

(h being the height MN and σ, the density of the vapour).

If a point L is taken inside the tube at the same level as N outside, the pressure at $L = P_1 + h\rho g$ ... (2)

ρ being the density of the solution.

If P is the osmotic pressure,

pressure at L − pressure at N = P

or $(P_1 + h\rho g) - (P_1 + h\sigma g) = P$

i.e., $P = hg(\rho - \sigma)$

or $hg = \frac{P}{\rho - \sigma}$ ... (3)

From (1) and (3) we get

$$P_2 - P_1 = P.\frac{\sigma}{\rho - \sigma} \quad \ldots (4)$$

and $P_2 - P_1 \propto P$ ... (5)

i.e., *the lowering of the vapour pressure of the solvent is proportional to the osmotic pressure.*

But, since the osmotic pressure is proportional to the concentration of the solution, c (measured in number of molecules of the solute per c.c. of the solution),

The lowering of the vapour pressure $\propto$ concentration, c

or $P_2 - P_1 \propto c$ ... (6)

The relation (3) can also be expressed in terms of the specific volume, v of the vapour. Neglecting σ in comparison with ρ, the relation (4) becomes $P_2 - P_1 = P.\frac{\sigma}{\rho}$ .... (7)

$$= \frac{P}{v\rho} \left(\because \sigma = \frac{1}{v} \right) \quad \ldots (8)$$

If v_0 is the specific volume of the vapour at the normal atmospheric pressure, P_0,

$P_0 v_0 = P_2 v$ (treating the vapour as a gas)

or $P_0\sigma_0 = P_2\sigma_0 \left(\because \sigma = \frac{1}{v} \text{ and } \sigma_0 = \frac{1}{v_0} \right)$

Substituting (7) for σ, $P_2 - P_1 = \frac{P}{\rho} \cdot \frac{P_2\sigma_0}{P_0}$

or $\frac{P_2 - P_1}{P_2} = \frac{P.\sigma_0}{P_0.\rho}$ ... (9)

The density of the vapour, σ in this treatment has been assumed to be uniform from the top to the bottom of the tube. This assumption, however, holds good when the length of the column is small, i.e., when the concentration and therefore the osmotic pressure are small.

[A relation which does not involve this assumption can be obtained as follows:

If p be the pressure of the vapour at a height h above the surface and σ the density of the vapour at that height, we have, for a small height dh, $dp = -g \cdot \sigma \cdot dh$ .... (10)

($-$ v sign indicates that the pressure decreases as h increases.)

Since $\frac{p}{\sigma} = \frac{P_0}{\sigma}$ or $\sigma = \frac{p\sigma_0}{P_0}$

(where σ_0 is the density of the vapour at the standard pressure P_0, temperature remaining the same.)

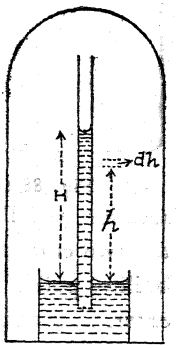


Fig. 88 (a)

$$dp = -g\frac{p\sigma_0}{P_0} \cdot dh$$

$$\text{or} \quad dh = -\frac{P_0}{g\sigma_0} \cdot \frac{dp}{p}$$

$$\text{and} \int_0^H dh = -\frac{P_0}{g\sigma_0} \cdot \int_{P_2}^{P_1} \frac{dp}{p}$$

$$\text{or} \quad H = \frac{P_0}{g\sigma_0} \cdot \log_e \frac{P_2}{P_1}$$

$$\text{Hence} \quad \log_e \frac{P_2}{P_1} = g \cdot \frac{\sigma_0}{P_0} \cdot H$$

Since, however, the osmotic pressure $P = \rho g H$,

$$\log_e \frac{P_2}{P_1} = \frac{\sigma_0 P}{\rho P_0} \quad \text{... (11).}]$$

(c) Elevation of Boiling Point

It is known that a liquid boils at a temperature at which its vapour pressure is equal to its external pressure and this temperature is called the boiling point. Thus, if the vapour pressure of a liquid becomes equal to the external pressure at T° A, the liquid boils at

T° A. Since the vapour pressure of the solution is less than that of a solvent, it follows that the vapour pressure of the solution will not be equal to the external pressure at T° A, the boiling point of the solvent. The solution must be raised to a temperature, say $(T+dT)°$ A, in order that its vapour pressure may be equal to its external pressure, i.e., the boiling point of the solution will be equal to $(T+dT)°$ A, a temperature higher than the boiling point of the solvent.

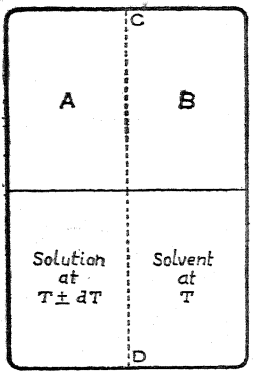


Fig. 88 (b)

This elevation of the boiling point of the solution over that of the solvent may be calculated from Thermodynamical considerations as follows :

Let A and B be the two compartments in a closed vessel separated by a semi-permeable membrane CD. Let the compartment A contain the solution and the compartment B, the pure solvent and the space above the solution in A and the solvent in B contain the vapour of the solvent. Imagine that the solvent is maintaned at T°A and the solution at $(T + dT)°$ A, so that the vapour pressure is the same in the two compartments. Then T°A and $(T + dT)°$ A, represent the boiling points of the solvent and the solution respectively for an equal external pressure.

Let the following cycle of operations be performed now :

(i) Let v c.c. of the solvent (v being the specific volume, i.e., the volume of 1 gm. of the solvent) be forced from B to A against the osmotic pressure P.

The work done in this process is equal to $P \times v$

(ii) Let the v c.c. of the solvent be now evaporated in the compartment A. If L_1 be the latent heat of the solvent at $(T + dT)°$ A, the energy absorbed, Q_1 in this process is equal to L_1 (since the mass of the v c.c. of the solvent is 1 gm.)

(iii) Let this vapour be transferred from A to B. There is *no expenditure of work in this process* since the vapour pressure is the same in A and B.

(iv) Let this vapour be now condensed back into the liquid state at T° A. If L is the latent heat of the solvent at T° A, the energy given out in this process Q_2, is equal to L.

This series of operation can be performed making dT as *small* as we please and the process itself can be carried out as *slow* as desired. It is seen that the cycle can be performed in the *reverse direction* also.

We will therefore have, from the **Second Law of Thermodynamics,**

$$\frac{\text{Work performed}}{\text{Heat given up}} = \frac{dT}{T} \quad \ldots\ (1)$$

i.e., $$\frac{Pv}{L} = \frac{dT}{T}$$

or $$dT = \frac{PTv}{L} = \frac{PT}{\rho L}$$ where ρ, the density of the solvent, is equal to $\frac{1}{v}$.

The elevation of the boiling point of the given solvent produced by 1 gm. molecule of any solute dissolved in 100 gm. of the solvent is called the **molecular elevation of the boiling point.**

(It is implied that in this case there is *no association or dissociation of the molecules* of the solute.)

(f) The elevation of the boiling point of a solution in the water as solvent may be calculated as follows:

The osmotic pressure produced by a solute in a solution can be calculated from the relation PV = RT (treating the osmotic pressure to be equivalent to the gas pressure.)

Suppose that a gram-molecule of oxygen (i.e., 32 gm.) is dissolved in 100 c.c. of water, concentration (as measured by number of gram-molecules per c.c. = $\frac{32}{100}$ gm. per c.c.

The volume, V, occupied by 1 gm. of the solute = $\frac{100}{32}$ c.c.

The gas constant, R is equal to $2 \cdot 596 \times 10^6$ ergs per degree (per gram). (As calculated from the gas equation PV = RT assuming the density of oxygen at N. T. P. to be 1·429 gm. per litre.)

Boiling point of water = 100° C = 373° A.

Hence substituting in the relation PV = RT, we have

$$P \times \frac{100}{8^2} = 2{\cdot}596 \times 10^6 \times 373$$

or $P = 3{\cdot}098 \times 10^8$ dynes per sq. cm.

i.e., the osmotic pressure produced by a gram-molecule of a solute in 100 c.c. of water is equal to $3{\cdot}098 \times 10^8$ dynes per sq. cm.

The elevation of the boiling point is then given by

$$dT = \frac{PT}{\rho L} = \frac{3{\cdot}098 \times 10^8 \times 373}{0{\cdot}9584 \times 540 \times 4{\cdot}182 \times 10^7}$$

$$= 5{\cdot}34°$$

(the density of water ρ at its boiling point being assumed from tables to be 0·9584 gm./c.c. and the latent heat of steam at 100°C, 540 cals. per gm. i.e., $540 \times 4{\cdot}182 \times 10^7$ ergs per gm.)

Now since the elevation of the boiling point depends only upon the concentration of the solute in the solvent, its value will be the same whatever solute may be in solution in water.

Hence the molecular elevation of the boiling point of water = 5·34° C.

This value is found to agree quite well with the results obtained from experiments, thus veryfying the theory of the osmotic pressure.

(g) Lowering of Freezing point

A similar treatment can be made in the case of fusion also. In the same Fig. 88 (b), let T°A represent the freezing point of the solvent in the compartment B and $(T - dT)$°A represent the freezing point of the solution, the external pressure in the two compartments A and B being the same. Let the following cycle of operations be now performed.

(i) Let 1 gm. of the solvent (occupying v c.c.) be forced from B to A.

The work done is equal to $P \times v$

(ii) Let the quantity be now frozen in A. The heat liberated is equal to L_1 (where L_1 is the latent heat of solidification at $(T - dT)$°A.

(iii) Let the solid be transferred from A to B. No work is

expended in the process since the external pressure is the same in the two compartments.

(iv) Let the solid now be allowed to melt, completing the cycle. The amount of heat absorbed is equal to L where L is the latent heat of fusion at T°A.

Then we have, as before, from the **Second Law of Thermodynamics**,

$$\frac{dT}{T} = \frac{Pv}{L} = \frac{P}{\rho L}$$

$$\text{or } dT = \frac{PT}{\rho L}$$

(h) The molecular depression of freezing point of water (i.e., lowering of the freezing point when a gram-molecule of any solute is dissolved in 100 gm. of the solvent) can be calculated as follows :

The osmotic pressure P is calculated from the relation $PV = RT$ or $P = \frac{RT}{V}$.

Now, $R = 2{\cdot}596 \times 10^6$ ergs per degree (per gram). Assuming that a gram-molecule (32 gm. of oxygen) is dissolved in water, the osmotic pressure P is given by

$$P = \frac{R \times T}{V} = \frac{(2{\cdot}596 \times 10^6) \times (273)}{(100/32)}$$

$$= 2{\cdot}27 \times 10^8 \text{ dynes/sq.cm.}$$

To calculate the lowering of the freezing point, we have,

$L = 80$ cal./gm. $= 80 \times 4{\cdot}182 \times 10^7$ ergs/gm.

P, the osmotic pressure $= 2{\cdot}27 \times 10^8$ dynes/sq.cm.

$T = 273°A$ and $\rho = 1$ gm./c.c.

$$\therefore dT = \frac{PT}{\rho L} = \frac{2{\cdot}27 \times 10^8 \times 273}{1 \times 80 \times 4{\cdot}182 \times 10^7} = 18{\cdot}4°$$

This value for the molecular lowering of the freezing point is the same for any solute dissolved in water and it has also been verified experimentally.

(i) **Molecular weight of a dissolved substance**

The molecular elevation of boiling point of water (or the molecular depression of the freezing point of water) is independent

of the solute dissolved in water. Thus determining the elevation of the boiling point (or the depression of the freezing point) for the given concentration of the solute in water, the molecular weight of the dissolved substance can be calculated. It is equal to that weight of the substance which, when dissolved in 100 gm. (i.e., 100 c.c.) of water produces an elevation of 5·34°C in the boiling point (or a depression of 18·4°C in its freezing point).

(j) Osmotic pressure of electrolytes

In the case of electrolytes (i.e., solutions which dissociate on passing an electric current through them), abnormally high results are obtained for the osmotic pressure and the elevation of the boiling point. It is explained that the molecules of the solute, in such cases, have undergone dissociation into positive and negative ions and the increase in the total number of such particles present is responsible for this behavior. If, for example, in certain solution, x gm. molecules of solute are dissolved in 100 gm. of water, the rise in the boiling point should be $(x \times 5{\cdot}34)$°C. provided there is no dissociation. If dissociation takes place, for every molecule that dissociates there will be two ions.

Let α be the degree of dissociation so that the number of molecules that are dissociated becomes αx and the corresponding number of ions $2 \alpha x$. The number of undissociated molecules is $(1 - \alpha) x$.

The total number of ions and molecules present in the solution $= 2 \alpha x + (1 - \alpha) x = x(1 + \alpha)$.

The observed elevation in the boiling point will have to be $5{\cdot}34\ x\ (1 + \alpha)$ instead of $5{\cdot}34\ x$; if the above reasoning is correct. From the values of the molecular elevation of the boiling point obtained in experiments on electrolytes, the degree of dissociation α was calculated. This value of α, so calculated, is found to agree with the value obtained from the measurements of electrical conductivity. The dissociation theory is therefore taken to explain the abnormal results for osmotic pressure and the elevation of the boiling points.

(b) In the case of colloids, abnormally low values for the osmotic pressure were obtained. This has been explained by

suggesting that there is association of some molecules of the solute in the solution so that the number of particles of solute is less than the number indicated by the chemical formula.

C—DIFFUSION IN GASES

The rate of diffusion in gases is also found to obey a law similar to Fick's law and the coefficient of diffusion is found to be different for different gases and also to vary with concentration to some extent.

145. Diffusion of gases on Kinetic Theory

Consider any plane MN in the mixture of two gases. Draw two planes AB and CD parallel to MN at a distance equal to the mean free path, λ, of the molecules in one of the gases in the mixture on either side of MN. The molecules can now travel straight from either of the a planes AB or CD to MN without changing their straight line paths due to collision with other molecules.

Let n_1 and n_2 be the numbers of molecules of one of the gases per c.c. at the planes AB and CD respectively.

We may imagine for simplicity that of the n_1 molecules per c.c. at AB, $\frac{1}{3}$ will be moving along the x-axis, $\frac{1}{3}$ along the y-axis and $\frac{1}{3}$ along the z-axis.

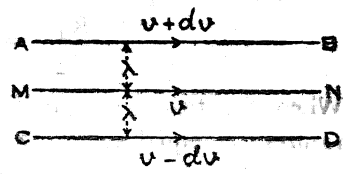


Fig. 88

Taking into consideration that the motion along any axis can be in the two opposite directions, we get the number of molecules per c.c. moving downwards from the layer AB to be equal to $\frac{n_1}{6}$.

Now, since the velocity with which a molecule moves is equal to c, the number of molecules that will pass in unit time across unit area of AB and reach MN will be equal to $\frac{n_1 c}{6}$.

Similarly, the number of molecules of the gas moving upwards from CD and reaching MN across unit area in unit time will be equal to $\frac{n_2 c}{6}$.

If n_2 is greater than n_1, the net transfer of molecules across MN in the upward direction $= \frac{(n_2 - n_1)}{6} c$ .... (1)

If m be the mass of a molecule, the net amount of the gas diffusing across MN per unit area per sec. $= m \frac{(n_2 - n_1)}{6} c$ .... (2)

Now according to Fick's law,

$$m \frac{(n_2 - n_1)}{6} c = \mathrm{R} \times \text{concentration gradient} \quad \ldots\ldots (3)$$

Difference in concentrations between AB and CD $= m(n_2 - n_1)$

The distance between them $= 2\lambda$

$\therefore$ Concentration gradient $= m \frac{(n_2 - n_1)}{2\lambda}$ .... (4)

$$\therefore \mathrm{R} \cdot m \frac{(n_2 - n_1)}{2\lambda} = m \frac{(n_2 - n_1)}{6} c$$

$$\text{i.e., } \mathrm{R} = \tfrac{1}{3} c \lambda \quad \ldots\ldots \quad \ldots\ldots (5)$$

146. Graham's law of diffusion

Now if R_1 and R_2 are the diffusivities of the two gases

$$\frac{\mathrm{R}_1}{\mathrm{R}_2} = \frac{c_1 \lambda_1}{c_2 \lambda_2} \quad \ldots\ldots \quad \ldots (6)$$

When the two gases are in equilibrium, their pressures and temperatures are equal so that $\lambda_1 = \lambda_2$

$$\therefore \frac{\mathrm{R}_1}{\mathrm{R}_2} = \frac{c_1}{c_2}.$$

But according to Maxwell, $\frac{c_1}{c_2} = \sqrt{\frac{c_1^2}{c_2^2}}$

$$= \sqrt{\frac{3p}{\rho_1}} \Big/ \sqrt{\frac{3p}{\rho_2}} = \sqrt{\frac{\rho_2}{\rho_1}}.$$

This is Graham's law: *The rate of diffusion of a gas is inversely proportional to the square root of its density.*

147. Effusion of gases

Effusion is the process whereby a gas escapes into vacuum from a thin-walled vessel through a small hole in it.

Graham carried out a number of experiments by allowing different gases to escape through short glass tubes and through small holes in thin brass plates. He found that the times of effusion of equal volumes of all gases were proportional to the square roots of their densities.

∴ *The rates of effusion (i.e., volumes escaping per sec.) are inversely proportional to the square roots of their densities.*

148. Explanation of Effusion on Kinetic Theory

Consider the effusion of a gas into vacuum through a small hole of area 'a' in a thin wall of an enclosed vessel.

The molecules which strike at the hole pass outwards and the number of molecules escaping through the hole per sec. $\propto$ 'a'.

$\propto \overline{c}$, the average velocity of the molecules.

$\propto n$, the number of molecules per c.c.

or the No. of the molecules passing out per sec. $\propto a.\ \overline{c}.\ n.$ (1)

It may be noted that the passage of the molecules into the vacuum will not be practically interfered with by the molecules travelling in the opposite direction since there is vacuum or the gas at very low pressure on the other side.

If m is the mass of a molecule,

mass of the gas escaping out per sec. $\propto a.\ \overline{c}.\ n.\ m.$ (2)

But $n.\ m. = \rho$, the density of the gas.

∴ Mass of the gas escaping out per sec. $\propto a.\ \overline{c}.\ \rho$ (3)

According to the kinetic theory of gases, the pressure of a gas, p, is given by the relation, $p = \frac{1}{3}\rho\overline{c^2}$

$$\therefore \overline{c^2} = \frac{3p}{\rho},$$

$$\text{Since } \overline{c} \propto \sqrt{\overline{c^2}},\quad \overline{c} \propto \sqrt{\frac{3p}{\rho}} \quad \ldots (4)$$

Hence the mass of the gas effusing out per sec.

$$\propto a.\ \overline{c}.\ \rho.$$

$$\text{i.e., } \propto a.\ \rho.\ \sqrt{\frac{3p}{\rho}} \quad \text{or} \quad = k.\ a.\ \sqrt{3p\rho}. \quad \ldots (5)$$

where k is a constant.

$\therefore$ The mass of the gas effusing out per sec. $=$

$$ka. \sqrt{3p\rho} \quad \ldots\ldots (6)$$

Now the efflux of two gases through the same hole may be studied.

$$\frac{\text{Mass of the first gas escaping out per sec.}}{\text{Mass of the second gas escaping out per sec.}} = \frac{k.\ a\sqrt{3p\ \rho_1}}{k.\ a\sqrt{3p\ \rho_2}} = \sqrt{\frac{\rho_1}{\rho_2}}$$

$$\therefore \frac{\text{Volume of the first gas escaping out per sec.}}{\text{Volume of the second gas escaping out per sec.}} = \sqrt{\frac{\rho_2}{\rho_1}}$$

Hence the rates of effusion of different gases / under similar conditions are inversely proportional to the square roots of their densities.

It follows from equation (5) that the mass of the gas effusing out per sec. $\propto \sqrt{3p\rho}$.

Since $p \propto \rho$ according to Boyle's law,

the mass of the gas effusing out per sec. $\propto \sqrt{p^2}$

and hence $\propto p$.

The rate of effusion of a gas is therefore proportional also to its pressure. In a mixture of gases, it follows that the proportion of the gases remains unaltered even after effusion since each gas effuses out in proportion to its partial pressure.

149. Transpiration

If the hole through which the gas escapes is fine but not short as in the case of unglazed procelain, the flow of the gas through it is called **transpiration** (or real diffusion by some authors). Graham investigated the factors which govern the process of transpiration and found that the rate of transpiration is:

(1) proportional to the difference of pressure on the two sides of the partition;

(2) inversely proportional to the square root of its density; and

(3) independent of the passage of another gas through the partition simultaneously.

150. Experiments on transpiration

A dry porus-pot A is fitted with a one-holed rubber stopper through which passes a long glass tube BC. The porus-pot contains air. It is inverted into a beaker containing some coloured water so that the end of the tube C is well under the water.

If now another beaker D containing hydrogen is inverted over the porus-pot, it is found that air from inside A escapes out through the water in the beaker. This is because hydrogen is lighter than air and more of hydrogen from D passes into A, than the air which passes out from A to D. The pressure inside A, consequently becomes greater than that outside and the gases inside A, therefore escape out in the form of bubbles through the water below. But after a time the pressures inside and outside A, become equal and the bubbles disappear.

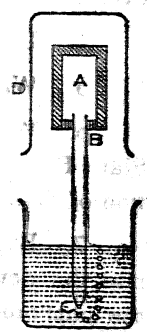


Fig. 89

If now the beaker D is removed, more of hydrogen from A passes out than the air which enters it from outside. The pressure inside A consequently becomes less and the water in the beaker rises up the tube BC. After all the hydrogen escapes out of A, the water in BC falls to its original level.

2. Air is passed through a tube AB which is the stem of a clay tobacco pipe. It is contained within a wider tube CD which is continually exhausted by means of an air pump. When air is passed through the inner-tube from A to B its nitrogen diffuses more rapidly into CD than oxygen (since the ratio of densities of nitrogen to oxygen is as 14 : 16) and the air coming out at B becomes richer in oxygen. If this process is repeated by passing the air coming out at B through a series of tubes similarly arranged, air which is many times richer in oxygen is obtained.

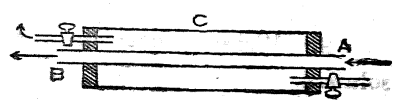


Fig. 90

The same method may be employed to separate partially any mixture of gases.

This process of separation of gases is called **Atmolysis.**

Exercises

1. Explain what you understand by diffusion.

2. Describe Graham's experiments on diffusion of liquids and explain the conclusions arrived at by him.

3. Define 'diffusivity'.

Explain how the diffusivity of a salt can be determined experimentally.

4. Deduce Graham's law of diffusion from kinetic theory.

5. Write notes on : 'effusion' and 'transpiration.'

6. What is the difference between diffusion and osmosis? State Fick's law. Explain how interdiffussion is utilised in production of high vacuum. [*A. U. Sept., 1962*]

7. Explain how the osmotic pressure of a substance in solution may be determined and give a short description of the theory of the phenomenon. Derive the relation between the osmotic pressure of a solution and the difference in temperature between the freezing points of the solution and the solvent. [*M. U. Sept., 1963*]

8. What is osmotic pressure and how is it measured? What are the laws governing the osmotic pressure of a dilute solution? Calculate the lowering in the freezing point of a solution having one gram-equivalent of a salt dissolved in one litre of water. [*M. U. Sept., 1964*]

9. Write a note on the elevation of the boiling point of a solution. [*M. U. April 1964*]

10. Write notes on osmotic pressure of electrolytes. [*M. U. April 1966*]

11. Write a note on diffusion in gases. [*A. U. April 1968*]

Chapter VIII

COMPRESSIBILITY

151. A uniform pressure, applied over a portion of matter of any kind, produces a change in its volume. *The compression produced is measured by the decrease in volume per unit volume of the substance,*

i.e., the compression $= \dfrac{\text{decrease in volume}}{\text{total volume of the substance}}$.

The compressibility (β) of a substance is defined as the ratio of the compression produced in the substance to the pressure, applied on it, which produces the compression.

Thus, if an increase of pressure dp produces a decrease in volume dv, in a substance having an initial volume v, the compressibility β of the substance is given by the relation $\beta = -\dfrac{dv \mid v}{dp}$.

(the $-$ *ve* sign indicates that the volume decreases as the pressure is increased.)

But the ratio of the pressure to the volume strain has been defined in the Chapter on 'Elasticity' as **the bulk modulus of the substance**, k.

$$\therefore k = \frac{dp}{dv \mid v}.$$

Thus *compressibility of a substance is seen to be equal to the reciproçal of its bulk modulus.*

$$\therefore \beta = \frac{1}{k} = -\frac{dv}{v \cdot dp}.$$

In the case of gases and liquids the ratio $-\dfrac{dp}{dv \mid v}$ is called the Elasticity of the gas or the liquid since gases and liquids cannot have any other strain than volume strain and it is the only modulus that they can have.

Matter in all the three states—solid, liquid and gaseous—will be subjected to compression when the pressure acting on it

is increased. But the amount of compression, produced for a given increase of pressure, depends upon the state of the substance. Gases are highly compressible; liquids are less compressible and solids are least compressible.

COMPRESSIBILITY OF SOLIDS

Solids can be subjected to three kinds of strains, namely, tensile, shear and bulk strains and they, consequently, possess the three moduli of elasticity — the Young's modulus Y, the modulus of rigidity n and the bulk modulus k. The three constants Y, n and k of a solid substance are related to one another by the relation, $Y\ (n + 3k) = 9\ nk$.

Since it is easier to determine Y and n, the bulk modulus k (and its reciprocal, β, the compressibility) is more often calculated by determining Y and n of the given substance, by any one of the methods elaborated in the Chapter on 'Elasticity' and substituting in this expression.

One of the methods of determining k for a solid by experiment is outlined below.

152. Determination of the bulk modulus

The bulk modulus of a substance can be determined directly by subjecting the given substance to a uniform pressure and measuring the compression per unit increase of pressure. But it is not convenient to measure the compression when the pressure is applied in all directions. A modified method due to **Amagat** is therefore adopted in which the body is subjected to a tensile stress in one direction and the corresponding change in volume measured.

The material to be studied (say glass) is taken in the form of a long cylindrical tube A. It is fitted with brass caps B and B_1 at its upper and lower ends respectively. The cap B is provided with a central hole in which is fixed a one-holed rubber stopper carrying a graduated and calibrated capillary glass tube C of known internal cross-section. The cap B is also provided with hooks (not shown) for suspending the tube while conducting the

experiment. The brass cap B_1, is provided with a hook H on its lower side which carries a weight hanger.

The whole of the tube AB and a part of the capillary tube C are filled with water *taking care to see that there are no air bubbles inside.* The initial reading of the level of water in C is noted. Weights are then added to the weight hanger in equal steps of M gm. Each time when a weight is added, it is found that the level of water in C falls, showing that the internal volume of the tube A has increased. The mean fall for the load of M gm. is determined.

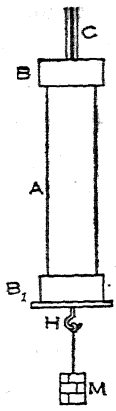


Fig. 91

If r be the radius of the bore of the capillary tube C and if l cm., be the fall in level when the load is increased by M gm. (i.e., when the force applied is Mg dynes) the diminution in volume is $\pi r^2 l$.

If V be the initial volume of the tube, the volume strain $= \frac{\pi r^2 l}{V}$.

The stress $P = \frac{Mg}{\pi (b^2 - a^2)}$ where 'b' and 'a' are the external and the internal radii of the tube A.

Now the stress which produces this strain is applied only in one direction. For calculating the bulk modulus we should know the volume strain when the same stress is applied equally in all the three directions. It has been shown in Art. 68 that the volume strain in the latter case will be 3 times the volume strain when the stress is applied only in one direction.

∴ The volume strain corresponding to a uniform dilatational stress $P = \frac{3 \pi r^2 l}{V}$.

∴ Bulk modulus of the material of the tube,

$$k = \frac{1}{\beta} = \frac{P}{dv / V} = \frac{Mg}{\pi (b^2 - a^2)} \Big/ \frac{3 \pi r^2 l}{V}$$

$$= \frac{Mg\, V}{3 \pi^2 r^2 l (b^2 - a^2)}.$$

152 (a). Determination of Poisson's ratio

Poisson's ratio of glass or the material of a cylinder can be determined by a slight modification of the experiment in Art. 152. A pin may be fixed horizontally to the base of the hollow cylinder and a travelling microscope capable of vertical movement focussed against the pointed end of the pin. The experiment is repeated as in Art. 152, noting the extension δl of the tube also, besides noting the increase in volume δv when a load is applied.

We know that the Poisson's ratio,

$$\sigma = \frac{\text{Lateral contraction}}{\text{Longitudinal elongation}}.$$

Let r be the radius of cross-section of the tube.

δr, the change in the value of r.

Let l be the length of the cylinder,

and δl be the increase in the length of the cylinder.

$$\text{Then } \sigma = \frac{\delta r / r}{\delta l / l} = \frac{l}{r} \cdot \frac{\delta r}{\delta l} \quad \ldots \quad \ldots \quad (1)$$

Area of cross-section $A = \pi r^2$, r being the radius of the tube.

$$\therefore \delta A = 2\pi r \cdot \delta r = 2A \cdot \frac{\delta r}{r} \quad \ldots \quad \ldots \quad (2)$$

Consequent upon the change in the radius and length, let the volume of the cylinder become $(v + \delta v)$.

$$(v + \delta v) = (A - \delta A)(l + \delta l)$$
$$= A \cdot l + A \cdot \delta l - \delta A \cdot l \text{ (neglecting } \delta A \cdot \delta l \text{ which will be very small)}$$

$$\therefore \delta v = A \cdot \delta l - \delta A \cdot l$$

$$\text{or} \quad \frac{\delta v}{\delta l} = A - \frac{\delta A}{\delta l} \cdot l$$

$$= A - \frac{2Al}{r} \cdot \frac{\delta r}{\delta l}. \quad [\text{from (2)}]$$

$$\text{Hence} \quad \frac{2Al}{r} \cdot \frac{\delta r}{\delta l} = A - \frac{\delta v}{\delta l}.$$

$$\text{i.e.,} \quad \frac{\delta r}{\delta l} = \frac{r}{2Al}\left(A - \frac{\delta v}{\delta l}\right) \quad \ldots \quad \ldots \quad (3)$$

Substituting for $\frac{\delta r}{\delta l}$ in (1) we get

$$\sigma = \frac{l}{r} \cdot \frac{r}{2Al} \left(A - \frac{\delta v}{\delta l} \right)$$

$$= \frac{1}{2} \left(1 - \frac{1}{A} \cdot \frac{\delta v}{\delta l} \right).$$

Thus knowing A, δv and δl, Poisson's ratio, σ can be determined.

COMPRESSIBILITY OF LIQUIDS

153. The determination of the compressibilty of a liquid is beset with the difficulty that increase of pressure on the liquid brings about a change in the volume of the container also. Experiments therefore give an apparent value. In order to obtain the true value of the compressibility of a liquid, an accurate knowledge of the bulk modulus of the containing vessel is also needed.

The vessel used to contain the liquid in these experiments is known as a **piezometer.** Regnault used a glass piezometer and studied compressibility of liquids up to a pressure of 10 atmospheres. Later, with improved methods, Bridgman was able to extend the range up to 12,000 atmospheres.

154. Regnault's apparatus consists of a cylindrical glass vessel A to which is connected a fine graduated capillary tube C. The piezometer, i.e., the vessel A, is surrounded by water or liquid contained in an outer vessel B for applying pressure externally on the piezometer.

It is preferable to keep the vessel B in another vessel D containing water to keep its temperature constant.

The pressure chamber is connected to the side tube T. Taps K, L, M and N (Fig. 92) serve to apply the pressure inside or outside the piezometer as desired.

1. When M and N are closed and K and L are open the *pressure inside and outside is equal to the atmospheric pressure.*

2. When L and N are closed and K and M are open the pressure is applied *internally only.*

3. When K and M are closed and L and N are open, the pressure is applied *externally only.*

4. When K and L are closed and M and N are open the pressure is applied *both internally and externally.*

When the piezometer is subjected to the same pressure internally and externally, not only is the liquid compressed but also the internal volume of the vessel decreases by an amount (say δv_1). *This value δv_1 will be equal to the diminution in volume that would be suffered by the vessel if it were solid and the same pressure be applied on it externally.*

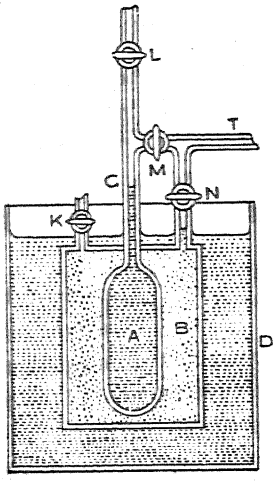


Fig. 92

If δv is the true diminution in the volume of the liquid, the observed diminution in volume δv_2 will be given by

$$\delta v = \delta v_1 + \delta v_2$$

$$\text{or} \quad \delta v_2 = \delta v - \delta v_1 \quad \ldots\ldots (1)$$

Now let k be the bulk modulus of the liquid,

K the bulk modulus of the solid vessel,

P the pressure applied,

and V the volume of the vessel.

Then $k = \dfrac{PV}{\delta v}$ or $\delta v = \dfrac{PV}{k}$.

Similarly $\delta v_1 = \dfrac{PV}{K}$

$$\therefore \delta v_2 = \frac{PV}{k} - \frac{PV}{K} = PV\left(\frac{1}{k} - \frac{1}{K}\right) \text{ [from (1)] } \ldots\ldots (2)$$

$$\text{Hence} \quad \frac{\delta v_2}{PV} = \frac{1}{k} - \frac{1}{K}$$

$$\text{or} \quad \frac{1}{k} = \beta = \frac{\delta v_2}{PV} + \frac{1}{K} \quad \ldots\ldots \quad \ldots\ldots (3)$$

Thus, knowing V, δv_2, P and K, the compressibility of the liquid is calculated. The value of K will have to be determined by an independent experiment. (Art. 152)

The glass piezometer used by Regnault was rounded at the lower end and therefore a correction had to be applied. But the correction applied was of doubtful value.

Later he used a strong piezometer with a flat bottom and studied the compressibility of different liquids up to 2,000 atmospheres.

155. Bridgman's Experiment

Bridgman used a strong chrome vanadium steel cylinder to contain the experimental liquid. The pressure on the liquid was increased by the advance of a piston of known cross-section. The apparent change in the volume of the liquid relative to the vessel was measured by the extent through which the piston advanced. The actual change in the volume of the liquid was then obtained by applying a correction for the expansion of the container. The temperature of the entire apparatus was maintained constant by a thermostatic control.

A coil of manganin wire (of resistance about 100 ohms) was inserted in the vessel for measuring the pressure applied. The resistance of a wire changes with increase of pressure and the relation between pressure and change of resistance of the manganin coil was determined by a separate experiment. The relation between resistance and pressure was found to be accurately linear up to a pressure of 12,000 atmospheres so that Bridgman used the relation to measure pressures up to 20,000 atmospheres without much error, by extrapolation.

The experiments on compressibility are beset with the difficulty of securing a leak-proof arrangement, especially when the pressures are very high. Bridgman used a special type of packing for securing the perfect leak-proof arrangement. But on finding the special type of packing to be effective only under high pressures, he used the ordinary packing also in addition to his special type so that the apparatus is perfectly leak-proof at high as well as relatively low pressures.

The packing used by him consists of a soft rubber packing B sandwiched between two mild steel or copper washers A and A. This arrangement is then held in position between a mushroom head D and a hard steel ring C. The head D is kept in contact with the liquid L while a piston rod P from above pushes the hard steel ring C so that the pressure is communicated to the liquid below D through all the intermediary layers.

In this arrangement, the thrust acting on all the different layers is the same. But since the area, over which the thrust acts on the rubber packing B in the form of a ring, is less, the pressure on it is more than on the head D and the liquid cannot leak out of it.

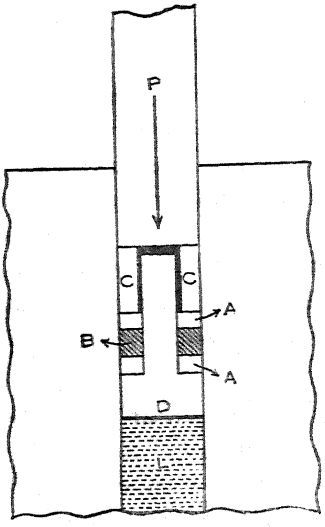


Fig. 93

156. Compressibility—Pressure and Temperature

The compressibility of solids is, in general, much less than that of liquids and gases. In all cases the decrease in volume under pressure may be the result of two effects: (1) a decrease in spacing between the molecules and (2) a decrease in the effective size of the molecules.

In the case of all liquids the compressibility decreases with the increase of pressure, very rapidly at first and slowly in the latter stages. The initial high compressibility may be due to the decrease in the spacing between the molecules while the slow decrease under high pressure may be due to the reduction in the effective size of the molecules.

The compressibility of solids and liquids increases, though slightly, with temperature. Water was found to be an exception to this in that its compressibility first decreases with temperature and then increases. It is found to have a minimum at about 50°C. The following relation summarises the results obtained by Bridgman for water.

At 0°C., $k = (2{\cdot}02 + 0{\cdot}000656\ P)\ 10^4$ and

at 50°C., $k = (2{\cdot}36 + 0{\cdot}000598\ P)\ 10^4$,

(where k is the bulk modulus of water and P is the pressure applied, expressed in kg. per sq. cm.)

COMPRESSIBILITY OF GASES

157. It is easily recognised that gases are highly compressible; a perceptible change in the volume of a gas can be produced by the application of even a moderate pressure.

Boyle's law gives the relation between the pressure and the volume of a given mass of gas at constant temperature. Such changes in pressure and volume which take place when the temperature of the gas remains constant are known as **isothermal changes.**

When a gas is subject to an isothermal change, the relation between pressure and volume is given by $PV = \text{constant}$ (i.e., by Boyle's law).

When a gas is compressed, work is done upon it and its temperature rises. When the experiment on verification of Boyle's law is conducted, some time is allowed for the gas to cool down to the temperature of the surroundings in order that the condition of *constant temperature* may be satisfied. In the process, the gas gives away some of its heat to the surroundings.

On the other hand, when a gas is allowed to expand, it does work and consequently it cools. Some time is allowed for the gas to gain heat from the surroundings so that the gas atains the same temperature as at the beginning and the condition of constant temperature is satisfied.

Thus in an isothermal change of a gas, some heat is either lost or gained by the gas.

158. Scientists have thought of another kind of change called **adiabatic change.** In an adiabatic change, the gas is not allowed either to give away or gain heat from the surroundings and the heat content of the gas remains constant. In an adiabatic compression, the gas rises in temperature and in an adiabotic expansion, the gas falls in temperature. The relation between pressure and volume of a gas which is subjected to an adiabatic change is given by $pv^{\gamma} = \text{constant}$.

(where γ is the ratio of the specific heat of gas at constant pressure and the specific heat of a gas at constant volume).

159. Elasticity of a gas

Let v be the diminution produced in a volume V of the gas by increasing the pressure by an amount P. Then $\frac{v}{V}$ is called the **Compression of the gas** and the Elasticity E of the gas is given by the relation, $E = \frac{P}{v/V} = \frac{PV}{v}$.

In the notation of calculus, $E = \frac{-dp}{dv/v} = -v\frac{dp}{dv}$.

(The $-ve$ sign indicates that the volume diminishes when the pressure is increased.)

The reciprocal of elasticity is called **compressibility**; it is the ratio of the compression per unit volume to the excess of pressure that produces it. It is denoted by the letter β.

$\therefore \beta$ = compressibility of the gas $= \frac{1}{E} = \frac{v/V}{P} = \frac{v}{PV}$.

In the notation of calculus, $\beta = -\frac{dv}{v} \Big/ dp$ or $\beta = -\frac{dv}{v.dp}$.

(a) Isothermal elasticity of a gas

For a gas at constant temperature, $pv =$ constant.

Differentiating, $dp \cdot v + p \cdot dv = 0$

or $dp \cdot v = -p \cdot dv$

or $-v\frac{dp}{dv} = p$.

$\therefore$ The isothermal elasticity, E, of a gas is equal to its pressure p, i.e., $E = p$.

(b) Adiabatic elasticity

For adiabatic changes in a gas, we have the relation

pv^{γ} = a constant

Differentiating $dp.v^{\gamma} + p.\gamma v^{\gamma-1}.dv = 0$

or $dp \cdot v = -p \cdot \gamma\, dv$

i.e., $-v\frac{dp}{dv} = \gamma p$.

The adiabatic elasticity of a gas is therefore equal to γp.

These relations hold for an ideal gas, i.e., a gas that obeys Boyle's law. Comparing the isothermal elasticity and the adiabatic elasticity, we find that the adiabatic elasticity is γ times the isothermal elasticity. γ, the ratio of the specific heats depends upon the gas and is equal to 1.66 for mono-atomic gases and 1.41 for air and so on. The experiments conducted on the compressibility of gases at constant temperature have already been described in Arts. 135 and 136.

Exercises

1. Define the terms 'compression' and 'compressibility'.

2. Describe an experiment to determine the compressibility of a solid supplied in the form of a cylindrical tube.

3. Describe Regnault's experiment on the determination of the bulk modulus of a liquid.

4. Write a note on Bridgman's work on the compressibility of liquids.

5. Write notes on:
'isothermal elasticity' and 'adiabatic elasticity'.

6. Determine an experiment to determine the compressibility of water. [*A. U. Sept., 1965*]

7. Write notes on compressibility of a liquid.
[*M. U. April 1966*]

Chapter IX

SURFACE TENSION

160. Illustration of the phenomenon

A rain drop falling to the ground is seen to assume a spherical shape. The drop as it falls is not subject to any constraining forces due to its weight. The question therefore arises as to what the forces are that make a falling drop spherical. Now *a sphere is the geometrical figure that has the minimum area of surface for a given volume.* It may therefore be concluded that the force that is responsible to give a spherical form to a falling drop is one that tends to make the area of its surface minimum. Whenever the effect of gravity is eliminated or reduced, the effect of this surface force becomes visible. Thus when small quantities of water are sprinkled on a greasy surface (on which water does not spread) they form into spherical drops. *The surface force tends to make the area of the surface minimum and consequently collect the water into a spherical drop.* The weight which depends upon the volume, on the other hand, tends to spread it. Since the quantity of water is very small, the effect of the latter becomes negligible and a spherical drop is formed under the action of the surface force.

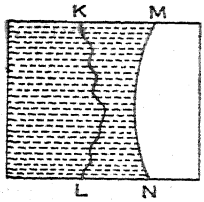


Fig. 94

A wire frame (Fig. 94) is made and two pieces of strings KL and MN are tied across the frame so that they are loose. Now the frame is dipped in soap solution and taken out. A soap film will then be formed in the frame and the pieces of string lie loose in the film. But on piercing the film between MN and the frame, this part of the film disappears and the thread MN will be drawn tightly to assume the form of an arc of a circle. This clearly shows that MN is acted on by forces which act at right angles to its length and which pull it so that the area of the film becomes minimum.

A circle is the geometrical figure which occupies the maximum area for a given perimeter. Since the string assumes a circular shape the area

which is cleared of the film is a maximum and the area covered by the film left over is a minimum.

These illustrations show that *the surface of a liquid is similar to a stretched elastic membrane, which tends to contract and occupy the minimum area possible.* It is found however that while the tension in a stretched membrane increases with the area, the tension in a liquid surface does not change with the area of the surface but is constant for the liquid, temperature remaining constant.

Definition : *Imagine a line drawn on the surface of a liquid. The force acting per unit length of the line and normal to it, tending to pull the surface apart along the line, is defined as the surface tension of the liquid.*

It is measured in dynes per cm. Since it is the ratio of a force to a length, its dimensions are $\frac{MLT^{-2}}{L} = MT^{-2}$.

161. The following are some more illustrations :

1. The property of surface tension is made use of in the manufacture of lead shot. Molten lead is poured down in a fine stream from a considerable height. The stream then breaks into a succession of small drops, as it falls, which cool and solidify on their way down. To preserve their spherical shape from damage they are caught in a deep basin of water.

2. If a small paint brush is dipped into water and taken out the hairs are found to cling together because the water films between hair and hair pull them together.

3. A new pin (which is not wetted by water) may be placed on a filter-paper, floated on the surface of water. After a time, the paper gets soaked in water. It is then gently pushed down so that it sinks. But the pin continues to float on water having a small depression under it. The surface of water supports the needle as if it were a skin.

4. The surface tension of water is lowered on contaminating it.

(a) If a paper boat, with a piece of camphor fastened to its stern, is floated on water, the boat moves forward.

The surface tension of pure water is greater than that of camphor water. The bow of the boat is therefore pulled by pure water with a greater force than its stern by the camphor solution. Consequently the boat is urged to move forward.

(b) The surface of stagnant water serves as a breeding place for mosquitoes, the surface tension of water supporting them on its surface. But when an oil is sprayed on the water, its surface tension is lowered and the mosquito breed sinks down and perishes.

162. Molecualr forces of attraction

Molecular forces are of two types (a) cohesive forces and (b) adhesive forces.

Forces of attraction between molecules of the same substances are called **cohesive forces.** The cohesive forces between any two molecules of substance probably vary inversely as the eighth power of the distance between them. Cohesive forces therefore become very large when the inter-molecular distances are very small. Thus they are very strong in solids, less in liquids and insignificant in gases.

Forces of attraction between molecules of different substances are known as **adhesive forces.** Molecules of water, for example, and the molecules of a glass container have forces of adhesion between them. These forces are different for different pairs of substances. Gum or glue has stronger adhesion with glass than water or alcohol with glass.

163. Molecular range of attraction and sphere of influence

The maximum distance up to which a molecule exerts a force of attraction on another is called **the range of molecular attraction.** It is very small, being of the order of 10^{-7} cm. The cohesive force of attraction decreases at a very rapid rate as the distance increases so that beyond a certain distance which is small, the force can be considered to be insignificant. But within that small distance they are very strong.

A sphere drawn with its centre coinciding with the centre of a molecule and radius equal to the range of molecular attraction, σ, is called **the sphere of molecular influence.** Obviously *a molecule,*

at the centre of the sphere of molecular influence attracts and is attracted in turn by the molecules present within the sphere. When the sphere of influence of a molecule is well within the substance, the molecule at the centre of the sphere will be on the average equally attracted in all directions and the resultant pull on it will be zero. Its velocity is therefore only that which depends on the temperature of the substance.

164. Work done in increasing the area of a surface

Suppose that a soap film (or any liquid film) is formed in a rectangular frame ABCD of which the side CD can slide. Let T be the surface tension and AB = CD = l.

The force on CD due to surface tension = 2 l. T (since the film has two surfaces). On account of this force the film tends to contract and in order to keep the film in equilibrium, an equal and opposite force, F, must be applied on CD also so that F = 2l T.

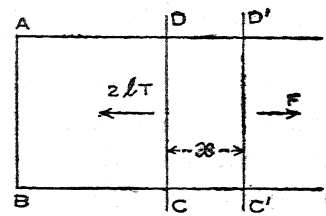


Fig. 95

If now the film is stretched so that CD moves parallel to itself to C'D' by a distance x,

work done by F in stretching = 2 l T x.

Now the increase in the area of the surface = 2 l x.

∴ Work done in increasing the surface by unit area
= 2 l T x / 2 l x = T.

Thus *the work done per unit area of the surface is numerically equal to the surface tension of the liquid.*

The work done against surface tension is stored up in the liquid as potential energy and the work done in increasing the area is sometimes called **surface energy.**

165. Explanation of surface tension on kinetic theory

Consider the molecules A, B, C, D and E near a liquid surface XY (Fig. 96). The circles round them indicate their respective spheres of influence.

The sphere of influence of the molecule A is wholly within the liquid. It does not therefore experience any resultant attraction in any direction.

The molecule B lies near the surface with a part of its sphere of influence (namely that above KL) outside the surface. If MN is a line parallel to KL and at the same distance below the centre as KL is above the centre, the molecule is symmetrically situated in the space KLMN. The resultant upward or downward force on B due to the molecules in the space KLMN vanishes since there will be as many molecules above B as there will be below it in that space. Considering now the spaces above KL and below MN, the space above KL contains much less molecules than that below MN because the former is vapour and the latter is liquid. Consequently, the pull exerted on B upwards will be very much less than that exerted downwards and the molecule B will be subjected to a resultant pull downwards.

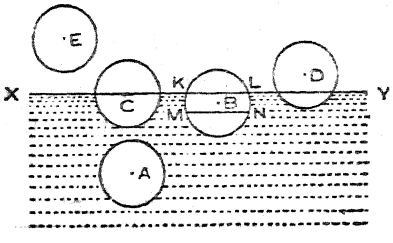


Fig. 96

The downward pull evidently increases as the space KLMN decreases and becomes maximum when MN coincides with KL, i.e., when the molecule is on the surface of the liquid as in the case of C.

When a molecule (say, D) is above the surface, it is similarly seen that the downward pull becomes less and, when its sphere of influence lies above the surface (as in the case of E) the pull completely vanishes and the molecule is free to wander about in the space above as a vapour molecule.

Thus the molecules which are within a small range above and below the surface of a liquid are subject to downward pull, the force being maximum on those which are on the surface. As a molcule moves from the interior towards the surface, it is therefore acted on by a strong force downwards which tends to prevent its escape from the liquid into the space above. This accounts for the *free surface* of a liquid.

166. Surface energy

If a molecule is to be brought from the interior of the liquid to the surface of the liquid, work will have to be done against the forces of attraction acting downwards, and the molecule consequently acquires some potential energy. Thus the surface of a liquid possesses potential energy on account of the molecules lying in it and whenever the area of the surface is to be increased by bringing more molecules to the surface, work will have to be done.

The potential energy of a system always tends to be a minimum and consequently, a liquid surface always tends to occupy a minimum area. Thus the *kinetic theory* successfully accounts for the phenomenon explained at the beginning of the Chapter.

167. Spreading of one liquid over another

Imagine a flat drop of a liquid A standing over another liquid B. Along the line of contact between A and B, there are three forces acting normal to the line :

(a) surface tension of the surface separating the liquid A and air, i.e., surface tension of A = T_2 ;

(b) surface tension of liquid B = T_1 ; and

(c) surface tension of the surface separating A and B = T_{12}.

Suppose now the drop spreads a little over the liquid B so that the area of contact between the liquids increases by an amount *ds*.

The consequent changes in the surface energy of the system will be as follows :

Increase in surface energy due to $T_2 = T_2 \times ds$

(since the area of contact between the drop and air increases.)

Increase in surface energy due to $T_{12} = T_{12} \times ds$ (for the same reason.)

Fig. 97

Diminution in surface energy due to $T_1 = T_1 \times ds$

(since the area of contact between the liquid B and the air diminishes by *ds*.)

Hence the net increase of surface energy

$$= T_2\, ds + T_{12}\, ds - T_1\, ds = (T_2 + T_{12} - T_1)\, ds.$$

Any system will tend to change in the direction in which its potential energy tends to decrease.

If now $(T_2 + T_{12} - T_1)$ is positive, the potential energy of the system will decrease when ds is negative. The liquid A therefore decreases in area and contracts into a drop which will stand on B. In this case, it will be possible to construct a triangle of forces with sides proportional to the three surface tensions T_1, T_{12} and T_2 respectively. A triangle so constructed with the three tensions respectively as sides is called **Neumann's triangle.**

If, however, $(T_2 + T_{12} - T_1)$ is negative, ds will increase in order that the potential energy may become minimum. The liquid A will then *spread over* the liquid B and form a thin layer. In this case it is not possible to construct Neumann's triangle because a triangle cannot be constructed when the sum of the two given sides is less than the third side.

Quincke and others established, by conducting experiments on a number of liquids, that in no case is it possible to construct Neumann's triangle. Thus, it is not possible to have a drop of one liquid remaining in equilibrium over another liquid. Sometimes, we may find apparent contradictions to this general observation. A drop of oil may be seen on water or a drop of water may stand in equilibrium on mercury. Careful observations reveal that, in the first case, oil has spread over water and is present there as a thin film. In the second case, the surface of mercury is found to be covered with grease when the water drop stands on it. Quincke showed that pure water spreads over pure mercury.

168. Angle of contact

In the case of solids, we cannot conceive of the surfaces remaining in a state of tension as in the case of liquid surfaces. However, it should be noted that surface tension of a liquid increases with fall of temperature even as it approaches the point of solidification and there is no reason to assume that the surface tension vanishes in the act of solidification. It will not be wrong therefore

to assume that solid surfaces are also associated with potential energy and that the potential energy per unit area of the surface may be called the surface tension of the solid in accordance with the relation established in Art. 164.

The equilibrium of a drop of liquid standing on the surface of a solid will now be considered.

Let AB be the drop of a liquid resting in equilibrium on a solid surface CD. Draw AE tangent to the surface of the drop at its point of contact, A, with the surface. Then $\angle$ EA is called **the angle of contact** and it is denoted by θ.

Since the drop is in equilibrium, the resultant of all the forces acting on it when resolved in any direction should be zero.

Let T_1, T_2 and T_{12} be the surface tensions for the solid-air, liquid-air and solid-liquid interfaces. Assuming that no other forces act on the drop and resolving the forces in a horizontal direction parallel to AB,

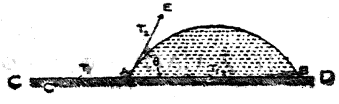


Fig. 98

$$\text{we have } T_1 - T_2 \cos\theta - T_{12} = 0$$

$$\text{or } T_2 \cos\theta = T_1 - T_{12}$$

$$\text{or } \cos\theta = \frac{T_1 - T_{12}}{T_2}.$$

If $T_1 > T_{12}$, $\cos\theta$ is positive and θ lies between 0° and 90°. If $T_1 < T_{12}$, $\cos\theta$ is negative and θ lies between 90° and 180°; this happens to be the case when mercury stands on glass, θ being about 140°.

It may be noted that in this case, the relative values of the surface tensions are such that the sum of any two of them is greater than the third and that Neumann's triangle can be constructed. However, if any one of the three surface tensions is greater than the sum of the other two, equilibrium is not possible and the liquid spreads on the solid surface. *The angle of contact of liquids which wet the surface and spread in this way, as water on glass for example, is zero.*

169. Measurement of angle of contact of mercury.

The angle of contact between mercury and glass can be determined by two simple experiments as follows:

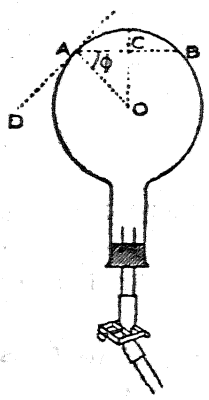


Fig. 99

(a) A round-bottomed flask, fitted with a one-holed rubber stopper through which passes a glass tube, is filled with clean mercury. A rubber tube is slipped over the glass tube and closed by means of a screw-clip. The flask is then inverted and supported on a stand.

Mercury is slowly drawn out until the surface of mercury inside the glass flask is plane, right up to the glass.

After adjusting the mercury surface carefully, the diameter of the mercury surface AB and of the flask ($=2 . AO$) are measured with a pair of calipers.

If $\angle CAO = \phi$, $\cos \phi = \frac{AC}{AO}$ from which ϕ is calculated.

ϕ, the angle of contact $= \angle DAC = 90 + \phi$.

(b) A strip of plane glass plate AB is dipped into mercury contained in a wide glass vessel. Then, meniscus is seen to form on either side of the glass plate. By suitably tilting the glass plate to one side mercury can be made to meet the plate without any curvature on that side as in Fig. 100. Using a plumb line and a scale, AC and DC are measured.

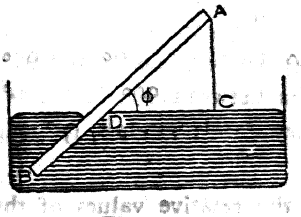


Fig. 100

If $\angle ADC = \phi$, $\tan \phi = \frac{AC}{DC}$ from which ϕ is obtained.

ϕ, the angle of contact $= \angle BDC = (180 - \phi)$.

170. Relation between pressure, curvature and surface tension.

Whenever the surface of a liquid is curved, as in a spherical drop or bubble, the surface tension of the liquid would, in tending to minimise the surface area, also tend to compress the liquid in the drop or the air in the bubble.

Consequently, the pressure inside the drop or bubble will be greater than the pressure outside. The excess of pressure inside

the drop or bubble depends upon its radius of curvature and the surface tension of the liquid.

(a) **Spherical drop**: Imagine a spherical drop of a liquid of radius r.

Let T be the surface tension of the liquid and P the excess pressure inside the drop.

Consider the drop to be divided into two hemispheres by a great circle ABC drawn on it in a vertical plane. Let ABCD be one of them.

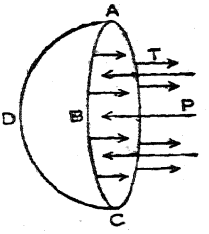


Fig. 101

The hemisphere ABCD is now subject to two forces in the horizontal direction:

(1) The thrust on the plane face ABC due to the excess pressure exercised by the other half on ABCD which is equal to $\pi r^2 P$ and which acts from right to left,

(2) the force due to the surface tension which acts along the circumference of the circle ABC, which is equal to $T.2\pi r$ and which acts from left to right.

Since the drop is in equilibrium, the two forces are equal.

$$\therefore \quad \pi r^2 P = 2\pi r.T$$

$$\text{or} \quad P = \frac{2T}{r}.$$

(b) **Spherical bubble**: In the case of a spherical bubble, there are two surfaces—the inner and the outer. The skin of the bubble is usually very thin so that the inner and the outer radii of curvature of the surfaces may be considered to be equal.

Consider again the equilibrium of a hemisphere as above. The thrust due to the excess pressure, P, exercised by the other half $= \pi r^2 P$ and

the force due to the surface tension acting on the surface $= 2.2\pi r T = 4\pi r T$.

$$\therefore \quad \pi r^2 P = 4\pi r T \quad \text{or} \quad P = \frac{4T}{r}.$$

(c) **Cylindrical drop**: Imagine a portion of a cylindrical drop contained between two horizontal planes ABC and A'B'D' at right algles to its axis,

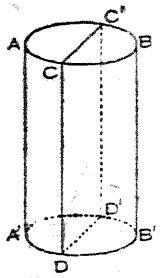


Fig. 102

Let r be the radius of cross-section of the cylinder, T, the surface tension,

l, the distance between the planes, i.e., the length of the cylinder ABB'A' and P, the excess pressure inside the drop.

Suppose that the cylindrical drop is divided into two halves by a vertical plane CC'D'D passing through the axes. Considering the equilibrium of one of the two halves CDB'B C'D', the forces acting horizontally on it are

(1) the thrust on the face CDD'C' due to the excess of pressure of the liquid exercised by the other half $= \frac{1}{2}P \times 2r \times l$ (acting from left to right),

(2) the force due to surface tension acting on the edges CD and C'D' $= T \times 2l$ (acting from right to left).

Since the drop is in equilibrium, $P \times 2rl = T \times 2l$

$$\text{or} \quad P = \frac{T}{r}.$$

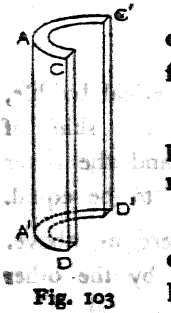


Fig. 103

(d) **Cylindrical bubble**: A cylindrical bubble consists of two surfaces, the inner and the outer. The forces acting horizontally on one half of the bubble are

(1) the thrust on the face CC'D'D due to the excess pressure inside the bubble $= P \times 2r \times l$ (acting from right to left)

(2) the force due to surface tension acting on the edges parallel to CD $= 2 \times T \times 2l$ (acting from left to right).

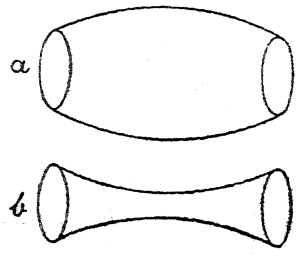
Since the bubble is in equilibrium, two forces are equal and $P \times 2rl = 2 \times T \times 2l$ or $P = \frac{2T}{r}$.

170. (a) General relation between pressure and surface tension in terms of the principal radii of curvature.

According to the principle of virtual work, when a body, acted on by a system of forces in equilibrium, suffers a small displacement,

the algebraic sum of the amounts of work done by these forces is zero. Applying this principle, the general relation between pressure and surface tension in terms of the principal radii of curvature of a surface can be determined.

For any element of a curved surface in general, two lines can be drawn on it which intersect each other at right angles and the radii of curvature of which are either maximum or minimum. These lines are called the *lines of principal curvature.* In some cases, the centres of the lines of curvature lie on the same side of the surface but in others, the centres lie on the opposite sides of the surface. Surfaces of the former type as in (*a*) such as those of spheres, ellipsoids etc., are called **Synclastic surfaces** while those of the latter types as in (*b*) such as the surface of a saddle on a horse back, the dice-box etc., are called **anticlastic surfaces.** The radii of curvature of these lines of principal curvature are called the *radii of principal curvature* of the surface.



a—Synclastic surface
b—Anticlastic surface
Fig. 104

(a) **General relation : Synclastic surface.** Let AB and BC represent the lines of principal curvature of a small element ABCD of a Synclastic surface and let O and O_1 be the centres of curvature of AB and BC respectively (Fig. 105).

Consider that the element is now displaced normally through a small extent x, so that the new positions of these lines of curvature are given by A_1B_1 and B_1C_1.

Let $AO = R$ and $BO_1 = R_1$.

Considering the similar triangles, A_1B_1O and ABO,

$$\frac{A_1B_1}{AB} = \frac{A_1O}{AO}$$

$$= \frac{(R + x)}{R}$$

$$\text{or } A_1B_1 = AB\left(1 + \frac{x}{R}\right).$$

Similarly, from the similar triangles, $B_1C_1O_1$ and BCO_1, we get

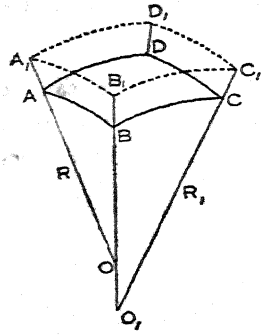


Fig. 105

$$\frac{B_1C_1}{BC} = \frac{B_1O_1}{BO_1} = \left(1 + \frac{x}{R_1}\right)$$

or $B_1C_1 = BC\left(1 + \frac{x}{R_1}\right)$

Area of the element before displacement $= AB \cdot BC$.

Area of the element after displacement $= A_1B_1 \cdot B_1C_1$.

$\therefore$ Increase in area of the element

$= (A_1B_1 \cdot B_1C_1) - (AB \cdot BC)$

$$= \left[AB \cdot BC\left(1 + \frac{x}{R}\right) \times \left(1 + \frac{x}{R_1}\right)\right] - AB \cdot BC$$

$$= AB \cdot BC \cdot x\left(\frac{1}{R} + \frac{1}{R_1}\right)$$

neglecting $\left[AB \cdot BC\left(\frac{x}{R} \cdot \frac{x}{R_1}\right)\right]$.

If T is the surface tension, the increase in the energy of the surface due to the displacement $= T \cdot AB \cdot BC \cdot x\left(\frac{1}{R} + \frac{1}{R_1}\right)$.

The pressure on the concave side of the surface is greater than that on the convex side and let P be the excess pressure on the concave side of the element.

$\therefore$ Work done by the pressure in the displacement of the surface $= P \cdot AB \cdot BC \cdot x$.

Now according to the principle of virtual work,

$$P \cdot AB \cdot BC \cdot x = T \cdot AB \cdot BC \cdot x\left(\frac{1}{R} + \frac{1}{R_1}\right) \text{ or } P = T\left(\frac{1}{R} + \frac{1}{R_1}\right) \quad \ldots (1)$$

If a film is considered instead of a liquid surface, the film has two surfaces and the increase in energy due to surface tension will be

$$2\,AB \cdot BC \cdot x\left(\frac{1}{R} + \frac{1}{R_1}\right)T$$

and $$P = 2\,T\left(\frac{1}{R} + \frac{1}{R_1}\right) \quad \ldots (2)$$

The excess pressure inside the surface of any particular type can be deduced from the above expression.

(a) *Spherical soap bubble*: In this case the radii of principal curvatures of the surface are equal so that $R = R_1$ and from (2)

$$\text{excess of pressure } P = 2T\left(\frac{1}{R} + \frac{1}{R}\right) = \frac{4T}{R}.$$

(b) *Spherical drop*: In this case also $R = R_1$ but the drop has only one free surface.

$$\text{Hence } P = T\left(\frac{1}{R} + \frac{1}{R}\right) = \frac{2T}{T}.$$

(c) *Cylindrical bubble*: In a cylindrical bubble one of the radii is equal to infinity, so that

$$P = 2T\left(\frac{1}{R} + \frac{1}{\infty}\right) = \frac{2T}{R}.$$

(d) *Cylindrical drop*: The cylindrical drop in addition to having one of the radii equal to ∞ will have only one free surface so that

$$P = T\left(\frac{1}{R} + \frac{1}{\infty}\right) = \frac{T}{R}.$$

(b) Excess of pressure inside an anticlastic surface

The centres of curvature O and O_1 of the lines of principal curvature AB and BC lie on opposite sides of the element ABCD as in Fig. 106.

Let the element ABCD be displaced normally to itself through a small distance x and let $A_1 B_1 C_1 D_1$ be its new position.

Now we have

$$A_1 B_1 = AB\left(1 - \frac{x}{R}\right)$$

$$\text{and } B_1 C_1 = BC\left(1 + \frac{x}{R_1}\right)$$

$\therefore$ Increase in area

$$= AB \cdot BC \cdot x\left(\frac{1}{R_1} - \frac{1}{R}\right)$$

Work done by the excess pressure $= P \cdot AB \cdot BC \cdot x$.

Work done against surface tension $= T\left[AB \cdot BC \cdot x\left(\frac{1}{R_1} - \frac{1}{R}\right)\right]$.

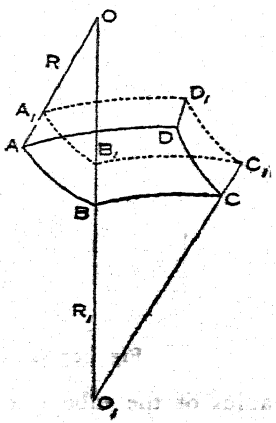


Fig. 106

By the principle of virtual work

$$P.AB.BC.x = T.AB.BC.x\left(\frac{1}{R_1} - \frac{1}{R}\right) \text{ or } P = T\left(\frac{1}{R_1} - \frac{1}{R}\right)$$

For a liquid film, the relation will be

$$P = 2T\left(\frac{1}{R_1} - \frac{1}{R}\right).$$

CAPILLARITY

171. When a capillary tube is dipped vertically into a liquid (such as water) contained in a beaker, the liquid is found to rise into the tube. All liquids which have an angle of contact less than 90° show a similar effect in a capillary tube. However, mercury whose angle of contact is greater than 90° shows a depression in a capillary tube dipped into it. The phenomenon of rise (or depression) of a liquid through a narrow tube is one of the most striking consequences of surface tension and is known as **capillarity**.

The ascent of kerosene or castor oil in the wick of an oil lamp, the rise of the sap in trees and plants and the absorption of ink in a blotting paper are some of the familiar examples of the same phenomenon.

DETERMINATION OF SURFACE TENSION OF LIQUIDS

172. Capillary rise method

Let a liquid (of density ρ) rise in a capillary tube which is dipped vertically in the liquid to a height h. Let r be the internal

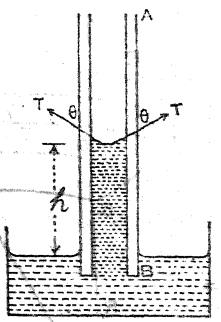


Fig. 107 (a)

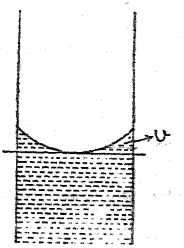


Fig. 107 (b)

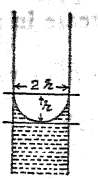


Fig. 107 (c)

radius of the tube (at the height h) and θ the angle of contact.

The forces acting on the liquid column are

(1) its own weight acting vertically downwards and

(2) the resultant upward pull due to the surface tension of the liquid, T.

If v be the small volume of the liquid in the meniscus, the volume of the liquid in the capillary tube $= (\pi r^2 h + v)$ [Fig. 107 (b)]

Weight of the liquid acting downwards on the column $= (\pi r^2 h + v)\, \rho\, g$. ... ... (1)

The vertical component of the force due to the surface tension $= T \cos \theta$ per cm. and it acts along a length, $2\pi r$.

$\therefore$ The resultant upward force due to surface tension $= 2\pi r\, T \cos \theta$. ... ... (2)

Since the column is in equilibrium, the two forces are equal.

$\therefore (\pi r^2 h + v)\, \rho\, g = 2\pi r T \cos \theta$ ... ... (3)

Hence, $T = \dfrac{(\pi r^2 h + v)\, \rho\, g}{2\pi r \cos \theta}$.

Case 1: For liquids, such as water, which wet glass, the angle of contact, $\theta = 0$ and the meniscus will be hemispherical.

The volume of water in the meniscus

$= (\pi r^3 - \frac{2}{3}\pi r^3) = \frac{1}{3}\pi r^3$ [Fig. 107 (c)].

$\therefore$ Surface tension, T, for liquids such as water which wet glass is given by the relation

$$T = \frac{(\pi r^2 h + \frac{1}{3}\pi r^3)\, \rho\, g}{2\pi r} = \frac{r\, \rho\, g\, (h + \frac{1}{3} r)}{2}.$$

If the radius of the bore is very small so that r is negligible as compared to h, $T = \dfrac{h\, \rho\, g\, r}{2}$.

Case 2: If the angle of contact is not zero, but the tube is fine so that v can be neglected

$$T = \frac{\pi r^2 h \rho g}{2\pi r \cos \theta} = \frac{r h \rho g}{2 \cos \theta}.$$

Case 3: If the angle of contact, $\theta = 90°$, there will be no capillary effect on the liquid.

Case 4 : If the angle of contact is greater than 90° so that cos θ is negative, there will be a capillary depression in the tube and the meniscus will be convex. This is the case with mercury, the angle of contact of which is 135° (with glass).

Experiment : A uniform capillary tube AB (about 20 cm. long) is cleaned with a strong solution of potassium dichromate to which concentrated sulphuric acid is added. It is then thoroughly washed with tap water. The tube AB can then be used directly if the experiment is with water ; otherwise the tube is to be dried before it is used.

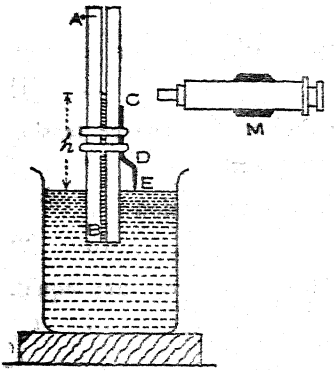


Fig. 108

The lower end of the capillary tube is introduced into the liquid, say water, contained in a beaker and the tube is clamped to a stand vertically. The liquid is found to rise in the tube to a certain height h. The tube is lowered by a cm. or two into the liquid and raised to the former position so that the walls of the tube above the meniscus are wetted with the liquid.

To determine h : A travelling microscope, provided with vertical traverse, is focussed on the meniscus and the reading s_1 on the vertical scale noted. The microscope is lowered and then focussed on the level of the liquid outside and the reading s_2 on the scale noted again. (It is, however, preferable to fasten a bent pin CDE to the tube and adjust the tip of the pin E to coincide with the surface of the liquid in the beaker before commencing experiment as in Fig. 108. The microscope may then be focussed on E and the reading s_2 noted.) The difference between the reading s_1 and s_2 gives 'h'.

To determine r : Strictly, 'r' should be measured at the point where the meniscus stands. But, since the tube is assumed to be of uniform cross-section, the diameters of the bore at the ends of the tube are measured using the travelling microscope and the mean value is taken as $2r$.

The density of the liquid at the temperature at the time of the experiment is determined separately. The surface tension T of the liquid is then calculated from the relation $T = \frac{(h+\frac{1}{3}r)\ P\ g\ r}{2}$.

The temperature at the time of the experiment is also recorded since *the surface tension of a liquid varies with temperature.*

It may be noted that the value of T is beset with the following uncertainties :

1. The angle of contact of the liquid with glass may slightly differ from zero.

2. The temperature of the liquid at the meniscus inside cannot be ascertained accurately.

3. Further, the contamination on the sides of the tube might not have been altogether eliminated in spite of thorough washing.

173. Method of drops

Let the flow of a liquid through a narrow vertical tube (3 or 4 mm. in diameter) be so regulated that the liquid falls in drops from the lower end of the tube. Each drop gradually grows in size and attains a limiting maximum volume before it detaches itself.

Consider the equilibrium of the drop just before it is detached.

Let r be the radius of the drop (which is also equal to the external radius of the tube when the liquid wets glass) and T the surface tension of the liquid.

The forces acting across the cross-section AB, are

(1) the weight of the drop, mg, which acts vertically downwards

(2) the downward force due to the excess pressure, P, inside the drop which is equal to $\pi r^2 P$.

(3) the force due to the surface tension of the liquid which acts upwards along the circumference of the section AB and which is equal to $2\pi r T$.

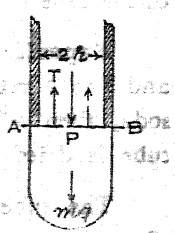


Fig. 109

Since the drop is in equilibrium,

the sum of the upward forces = the sum of the downward forces,

i.e., $2\pi r T = mg + \pi r^2 P$

Rayleigh assumed the drop to be cylindrical at the cross-section of the bore so that $P = \frac{T}{r}$.

$$\therefore 2\pi r T = mg + \pi r^2 \cdot \frac{T}{r}$$

or $$\pi r T = mg$$

and $$T = \frac{mg}{\pi r}.$$

But the fall of a drop is essentially a dynamic problem and a statical treatment of the same cannot be complete. Also sometimes, due to the unstable conditions, a small drop may also be seen to detach from the tube after the main drop falls. (This can be clearly seen if water is allowed to drop from the glass tube into kerosene oil or cocoanut oil contained in a beaker). Owing to these uncertain factors, the values for T obtained from experiments were found to be higher than the actual values.

Rayleigh suggested that if π were substituted by 3·8, the experimental results would agree fairly well with actual values. Thus we have $T = \frac{mg}{3{\cdot}8r}$.

[Sometimes the lower end of the tube is given a coating of wax so that the liquid does not spread sideways. In such a case, the value of the radius of the bore (and not of the external radius of the tube) will have to be taken for r.]

Experiment: A narrow glass tube (of about 15 cm. long and 3 to 4 mm. diameter) is cleaned thoroughly first with caustic soda, then with conc. nitric acid and finally with tap water. The tube is dried if the experimental liquid is other than water.

The tube is next connected to the nozzle of a funnel by means of a short rubber tube. A screw-clip provided on the rubber tube can be used to control the flow of the liquid down the glass tube. The liquid (say, water) is poured into the funnel and the flow of water down the glass tube is adjusted so that drops detach themselves from the lower end of the tube at the rate of one or two per minute.

A clean and dry beaker of mass m_1 is next introduced under the glass tube and sufficiently large number of drops, n (say, 100) is collected into the beaker. The mass m_2 of the beaker, together with the liquid collected, is found.

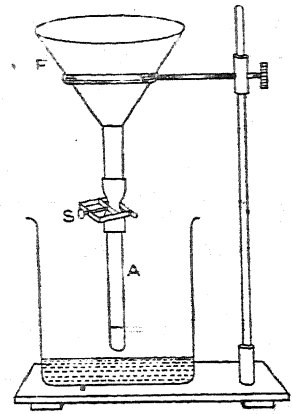


Fig. 110

The external diameter, $2r$, of the glass tube at the lower end is determined either by focussing a microscope or by using a vernier calipers.

The surface tension of the liquid, T, at the room temperature is then calculated from the relation

$$T = \frac{(m_2 - m_1)\,g}{3 \cdot 8\, r\, n} = \frac{mg}{3 \cdot 8\, r}$$

(m, being the average mass of a drop).

174. Interfacial tension between two immiscible liquids

When one liquid rests on another without mixing with it, the interface between the two liquids possesses energy just like the surface of a liquid. *The interfacial tension is the value of the force acting per cm. normal to a line drawn on the interface.*

The heavier liquid (say water) is taken into a funnel and it is allowed to flow down through a narrow tube into the second liquid (say kerosene oil). The flow is regulated so that drops of water detach themselves into kerosene one by one.

The weight of the drop is, in this case, reduced by the upthrust of the lighter liquid. If ρ and σ are the densities of water and kerosene respectively, the apparent weight of the falling drop (when formed in the lighter liquid) is equal to $\left[mg \left(1 - \frac{\sigma}{\rho} \right) \right]$, m being the actual mass of the drop.

Thus obtaining m and r as in the experiment by the method of drops, the interfacial tension T between the two liquids can be calculated from the relation,

$$T = mg\, \frac{(1 - \sigma/\rho)}{3 \cdot 8\, r}.$$

175. Torsion balance method

When a glass plate is held horizontally so that its bottom surface touches the free surface of water (or of a liquid which wets glass), the plate is pulled downwards by a force due to the surface tension of water. This force acts along the perimeter to the lower surface of the plate.

If l and t are the length and thickness of the lower surface respectively its perimeter is equal to $2\,(l + t)$.

Let T be the surface tension of water.

The pull F acting vertically downwards is given by
$$F = 2\,(l + t)\,T.$$

Determining F by using a torsion balance, T, the surface tension of the liquid can be determined experimentally.

Experiment: The torsion balance consists of a rigid U-shaped frame U fitted into a heavy tripod. The frame can be raised or lowered and also fixed at any desired height by means of a screw *s*. A fine steel wire, *w*, is stretched across the frame so as to be taut and it carries a metal blade at its mid-point at right angles to its length. The blade extends a little to one side and carries a sliding weight W. The blade on the other side, which serves as the twisting arm of the torsion balance, carries a scale-pan P, from which a metal frame *f* is suspended. A thin rectangular glass plate is inserted into the frame *f* for determining the vertical pull of the liquid. The end of the twisting arm is pointed and it moves over a scale S fixed to the frame of the apparatus itself. A levelling screw N provided on the leg of the tripod opposite to the glass plate can be used to tilt the whole apparatus and thereby raise or lower the glass plate.

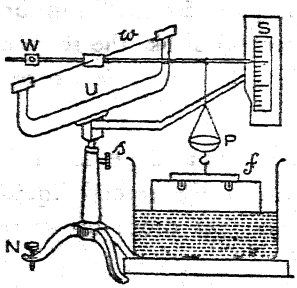


Fig. 111

Before commencing the experiment, the glass plate is cleaned well first with acidified potassium bichromate solution and then with tap water. A glass trough containing the experimental liquid is brought under the plate. *The lower edge of the plate is adjusted*

to be perfectly horizontal by seeing that the image of the edge becomes parallel to it.

The sliding weight on the blade and the U-shaped frame (if necessary) are adjusted so as to bring the lower edge of the glass plate very close to the surface of the liquid. The screw N is then worked in the proper direction so as to make the lower edge of the plate touch the surface of the liquid. If now the screw is worked slowly in the opposite direction, it will be found that the pointed end of the arm moves lower and lower down the scale and that at one stage, the glass plate gets detached from the surface of the liquid. *The reading on the scale is noted just before the plate tears itself away from the liquid.*

The experiment is repeated to get at least three consistent readings on the scale just before the plate leaves the surface of the liquid.

The trough is removed and *the plate dried with a piece of blotting paper.* Necessary weights are then placed in the scale-pan so as to make the pointed end of the arm coincide with the previous reading. If m be the mass of the weights added, the force F required to twist the wire to the required reading is equal to mg and it is equal to the force exerted by the surface tension on the lower surface of the plate.

The length l of the lower edge is measured with a slide calipers and its thickness t with a screw gauge.

The surface tension T of the liquid at the room temperature is then calculated from the relation,

$$2\,(l + t)\,T = mg \quad \text{or} \quad T = \frac{mg}{2\,(l + t)}.$$

175. (a) Surface tension of mercury: Quincke's drop method

The surface tension of mercury and its angle of contact can be determined by Quincke's method.

Theory: It has been explained in Art. 167 that smaller amounts of liquids, placed on horizontal surfaces, which are not wetted by them, collect as spherical drops. The effect due to surface tension in this case excels that due to gravity in giving the drop, a spherical

shape. If more liquid is added to the drop, the drop grows in size; the forces due to gravity preponderate and the drop flattens. When there is a sufficient quantity of liquid, the surface becomes horizontal and any further addition of liquid does not increase the height of the drop.

Let AB be the horizontal surface of a sufficiently big drop of mercury placed on a glass plate PQ (Mercury does not wet glass). The edges of the drop C and D are curved as in Fig. 111 (a). Let h_1 be the height of the drop and ρ the density of mercury.

The relation between the forces acting on such a drop can be established as follows:

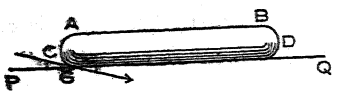


Fig. 111 (a)

Consider a thin slice of the drop AKMG having a small width KL (= dx) cut out of the drop by drawing two vertical parallel planes at a distance dx apart and another vertical plane KLNM, at right angles to them.

Let GT be the tangent plane drawn to the liquid surface along G, the line of contact of mercury with the glass plate.

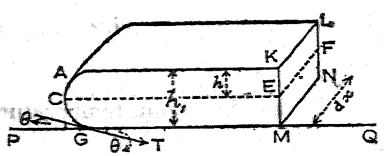


Fig. 111 (b)

Let $\angle TGM = \theta$, the supplement of the angle of contact. For mercury θ is about 40°. The forces acting on the thin slice are as follows:

(1) the hydrostatic thrust acting on the front face AKMG.

(2) the hydrostatic thrust acting on the face at the back, parallel to AKMG.

(3) the force due to surface tension acting on KL and perpendicular to it, equal to T.dx (towards right).

(4) The force due to surface tension acting at G = T. cos θ. dx acting towards right, and

(5) the hydrostatic thrust acting on the face KLNM due to the liquid lying to the right of it

= pressure at the centre of gravity of the fac KLNM × area of KLNM

$$= \left(\frac{h_1}{2} \times \rho \times g\right) \times \left(h_1 \times dx\right) = \frac{h_1^2 \rho g}{2} \cdot dx.$$

and it acts towards left.

The forces in (1) and (2) are at right angles to those in (3), (4) and (5) and are therefore in equilibrium between themselves. Similarly, forces (3), (4) and (5) are in equilibrium.

$$\therefore T.\, dx + T.\cos\theta \cdot dx = \frac{h_1^2 \rho g}{2} \cdot dx.$$

$$\text{or} \quad T(1 + \cos\theta) = \frac{h_1^2 \rho g}{2}.$$

The surface tension of mercury or its angle of contact can be determined if one of them is known, using a travelling microscope to measure h_1.

It can be seen from the figure that the direction of the tangent drawn to the surface at a point on the edge ACG depends upon the point where the tangent is drawn. As the point is taken higher up towards A, the tangent inclines more and more towards the vertical and it becomes exactly vertical at some point C. The forces due to surface tension at this point C act in the vertical direction. If, therefore, a horizontal plane CEF, parallel to AB and passing through C is taken and the equilibrium of the slice above this plane is considered.

The above relation reduces to $T = \frac{h^2 \rho g}{2}$,

(where h is the height, KE of the drop above the plane CEF) since θ becomes 90° in this case.

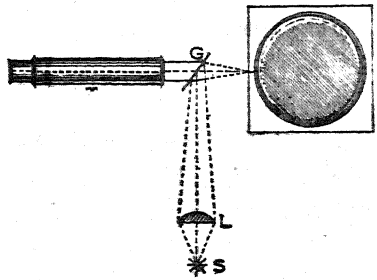
This relation enables us to determine the surface tension of mercury even if the value of θ is not known. But the point C on the curved end where the tangent is vertical cannot accurately be located. To get over this difficulty, Edser modified the objective of the reading microscope and used it to determine the surface tension of mercury as described in the next Article.

Experiment : Edser fitted plane glass plate G in the objective of a travelling microscope as in the Fig. 111 (c). Rays from a source

of light S are made to fall on the glass plate G by means of a condensing lens L. The rays reflected from the plate then illuminate the edge of the mercury drop. The illuminated edge can be seen through the miscroscope by means of the rays entering it through the glass plate G.

All points such as C on the edge of the drop Fig. 111 (b), where the tangents are vertical, lie on a bright horizontal line which can be seen through the microscope when focussed.

To determine h: The microscope is focussed on the bright line passing through C and moved up and down so that the horizontal cross-wire coincides with the line. The reading on the vertical scale (s_1) is noted. The microscope is moved up and set so that the horizontal cross-wire coincides with the flat top surface of the drop. Some fine dust such as lycopodium powder may be scattered lightly on the top surface of the drop to facilitate the proper setting of the microscope. The reading, s_2, is taken on the vertical scale. The difference between the two readings gives h.



(As seen from above)
Fig. 111 (c)

The surface tension T of mercury is then calculated from the relation $T = \frac{h^2 \rho g}{2}$ (ρ being the density of mercury and g the acceleration due to gravity).

To determine the angle of contact: The microscope is lowered and set with its horizontal cross-wire against the bottom of the drop. The reading on the scale, s_3 is noted. The height h_1 is then equal to $(s_2 - s_3)$ (The height h_1 of the drop can also be determined using a spherometer).

Knowing T from the above experiment and determining h_1, the angle θ is calculated from the relation $T = \frac{h_1^2 \rho g}{2(1+\cos\theta)}$

The angle of contact of mercury is equal to $(180° - \theta)$.

VARIATION OF SURFACE TENSION WITH TEMPERATURE

176 The surface tension of liquids decreases with rise of temperature except in the case of molten cadmium and copper. The variation may be studied in several ways.

(a) **U-tube method :** A small U-tube, consisting of a capillary tube AB on one side and a tube CD having a diameter of about 1 cm. on the other, is used for studying the variation of surface tension of a liquid with temperature.

The experimental liquid is added to the wider tube so that it fills the bend and rises into the two limbs. It will be found that the level in the capillary tube is higher than that in the wider tube.

Let T be the surface tension of the liquid and let r_1 and r_2 be the radii of the bores of AB and CD respectively. For a liquid which wets glass and the angle of contact of which is 0°, the pressure just below the surface of the liquid in $AB = H - \frac{2T}{r_1}$.

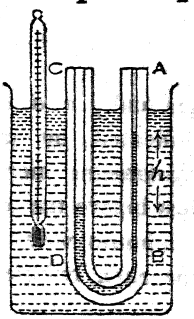


Fig. 112

(H being the atmospheric pressure). Similarly the pressure just below the surface of the liquid in $CD = H - \frac{2T}{r_2}$.

∴ The difference in pressures just below the surfaces of the liquid in CD and AB $= 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

If h is the difference in levels of the liquid surfaces in AB and CD respectively and ρ, the density of the liquid, the difference in pressures is equal to $h \rho g$.

Hence, $2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = h \rho g$.

Thus measuring h, r_1, r_2 and knowing ρ at the temperature of the experiment, the surface tension of the liquid can be determined at different temperatures.

(b) **Jaegar's method:** A large vessel A is closed with a two-holed rubber stopper through which are passed two glass tubes. One of them is connected to a reservoir of water theough a tap B and the other is joined through a tap C to a manometer M and a vertical tube DE. The tube DE ends in a narrow orifice at E and dips into the experimental liquid contained in a beaker. The beaker can be heated so as to raise the temperature of the liquid to any desired value.

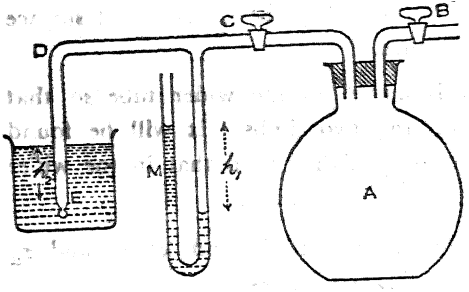


Fig. 113

As the water from the reservoir is allowed to slowly drop into A, the air in the vessel is displaced. The displaced air forces its way through the tube DE and escapes in the form of bubbles through the liquid. Each bubble is found to grow gradually and as it grows, the pressure inside the tube rises as indicated by the manometer. The difference in levels in the manometer increases as the bubble grows, attains a maximum value just before the bubble is detached from the nozzle and drops to a minimum value after the bubble breaks off. This process is repeated for each bubble.

The maximum value of the pressure of the air inside is obtained by noting the lowest point to which the manometric liquid falls in one limb and the highest point to which it raises in the other limb when each drop is detached.

If h_1 is the maximum difference in the levels of the liquid in the two limbs, H, the atmospheric pressure and ρ_1 the density of the manometric liquid, the maximum pressure inside the bubble $= H + h_1 \rho_1 g$ ...(1)

If h_2 is the depth of the orifice E below the free surface of the liquid in the beaker and ρ_2 the density of the liquid in it, the pressure just outside the bubble at E $= H + h_2 \rho_2 g$ ... ... (2)

Now the excess of pressure, P, inside the air bubble in the liquid over that outside is equal to

$$(H + h_1 \rho_1 g) - (H + h_2 \rho_2 g) = (h_1 \rho_1 - h_2 \rho_2) g$$

i.e., $P = (h_1 \rho_1 - h_2 \rho_2) g$ .... (3)

But it has been shown that the excess of pressure inside a spherical bubble of any liquid is equal to $\frac{4T}{r}$, T, being the surface tension of the liquid and r, the radius of the bubble. Since the air bubble has only one surface of contact with the liquid in this case, the excess pressure over that outside is equal to $\frac{2T}{r}$ only,

i.e., $P = \frac{2T}{r}$ .... (4)

From (3) and (4), $P = \frac{2T}{r} = (h_1 \rho_1 - h_2 \rho_2) g$ ... (5)

or $T = \frac{(h_1 \rho_1 - h_2 \rho_2)}{2} g r.$

(It may be noted that as the air forces out its way at E forming a bubble, the radius of curvature decreases gradually as the bubble grows and the pressure inside the bubble increases. When the bubble is in the form of a hemisphere having for its radius, the radius of the orifice, the radius of curvature attains the minimum value and the pressure, the maximum value. When a bubble passes this stage, it becomes unstable because the radius increases and the pressure inside due to the surface tension should fall. The bubble therefore grows very quickly and detaches itself from the orifice. The process begins once again for the next bubble.)

The experiment does not give correct values of T as these conditions are not entirely static and as the radius of the bubble just when it breaks away cannot be ascertained with certainty. *This method may therefore be conveniently used to compare the surface tensions of different liquids or the surface tensions of the same liquid at different temperatures.*

(c) The variation of surface tension with temperature may also be studied using a torsion balance.

177. Surface tension and temperature

Surface tension decreases with temperature and for small ranges of temperature, the variation is given by the relation $T_t = T_o (1 - \alpha t)$.

(T_O being the surface tension of the liquid at 0°, T_t the surface tension at t°C and α the coefficient of surface tension.)

The surface tension of any liquid becomes zero at its critical temperature and the interface between vapour and liquid at that temperature disappears. The absolute value of the surface tension may therefore be supposed to depend upon the remoteness of its temperature from the critical temperature.

Van der Waals suggested the following formula for the variation of surface tension with temperature :

$$S_T = S_o \left(1 - \frac{T}{T_o}\right)^{\frac{3}{2}}$$

(where S_T and S_o are the surface tensions respectively of the liquid at T°A and 273°A and T_o is its critical temperature.)

178. Rise of a liquid between two parallel plates

If two plane glass plates are kept close to each other in water (or in any liquid which wets glass) the water (or the liquid) is seen to rise between them.

The *meniscus between the plates will be cylindrical* in form, the diameter of the cross-section of the cylinder being equal to the distance, d, between the plates.

Now the excess of pressure above the meniscus over that just below $= \frac{T}{r}$. But $r = \frac{d}{2}$.

$\therefore$ The excess of pressure $P = \frac{2T}{d}$.

The pressure at a point just below the meniscus $= \left(H - \frac{2T}{d}\right)$ where H is the pressure of the atmosphere.

If h is the height to which the water rises between the plates, the hydrostatic pressure due to the rise of the liquid $= h \rho g$, so that

the pressure at a point just below the meniscus is equal to $(H - h\rho g)$.

$$\therefore \frac{2T}{d} = h\rho g, \quad \text{or } h = \frac{2T}{d\rho g}.$$

The rise of the liquid between the plates is therefore inversely proportional to its density and also to the distance between the two plates.

179. Drop of a liquid between two plates

Two clean glass plates KL and MN kept face to face can be pulled apart with least effort. But if a small drop of water (or any liquid) is squeezed into a thin film between them, a considerable effort is required to separate them. The surface tension of water causes a difference of pressure between the water and the air outside.

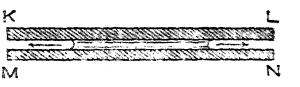


Fig. 114

Since the meniscus at the edge of the film assumes a cylindrical form, the excess of pressure of the air outside over that of the liquid is equal to $\frac{2T}{d}$ (Art. 178).

If A is the area of film, the force due to the excess of pressure $= A \times \frac{2T}{d} = \frac{2TA}{d}$.

Since d is very small, this force with which the plates are pressed together becomes quite considerable.

Problem : Two plane glass plates are separated by a thin film of water of 0·1 mm. thickness. If the area of each plate wetted by the film is 10 sq. cm., find the minimum force required to separate the two plates.

(Surface tension of water = 75 dynes/ cm.)

Force pressing the plates together $= \frac{2T}{d} \cdot A$.

Area of each plate wetted, $A = 10$ sq. cm.

Distance between the plates, $d = 0{\cdot}01$ cm.

Surface tension of water, $T = 75$ dynes/ cm.

$$\therefore \text{Force} = \frac{2 \times 75}{0{\cdot}01} \times 10 = 1{\cdot}5 \times 10^5 \text{ dynes.}$$

Hence the force required to separate the plates should at least be (1.5×10^5) dynes.

180. Shape of a drop on a horizontal surface

The forces acting on a drop of a liquid placed on a horizontal surface (which it does not wet) are (a) the force due to surface tension and (b) the force due to gravity due to its own weight.

The force due to surface tension causes a drop to assume a spherical shape while the force due to gravity tends to flatten the drop and spread it out until it becomes a thin film. Thus a small drop of negligible weight assumes a perfectly spherical shape while a large drop takes a form intermediate between a sphere and a thin film under the combined action of the two forces.

181. Evaporation at curved surfaces: Formation of clouds

Kelvin has deduced an interesting relation for the equilibrium vapour pressure of a liquid when its surface is curved.

Let AB be a capillary tube of radius r introduced vertically into a liquid contained in an enclosure which has been exhausted of air. After a time, steady conditions will be obtained in the enclosure. The liquid rises into the tube and the space above the liquid in the enclosure will be filled with the vapour of the liquid.

Let the liquid rise to a height h in the capillary tube and M be a point in the curved meniscus. Further, let N be a point on the surface of the liquid outside the tube, and Q a point in the vapour vertically above N and at the same level as M.

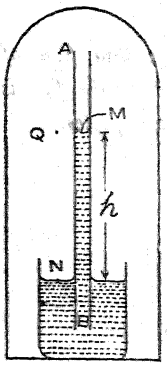


Fig. 115

Then the pressure of the vapour p just above M is the same as at Q. If P is the pressure of the vapour at N on the surface of the liquid, and σ is the density of the vapour in the enclosure, the pressure at Q = $P - h\sigma g$.

$\therefore$ The vapour pressure, p, above the concave surface of the liquid at M = $P - h\sigma g$.

i.e., $p = P - h\sigma g$ ... ... (1)

Since the liquid at N and the liquid at M are in equilibrium with the vapour above them in the respective positions, the vapour

pressure P at N is the equilibrium vapour pressure over the plane liquid surface there, and the vapour pressure P at M is the equilibrium vapour pressure over the concave liquid surface at M.

Now P is the equilibrium vapour pressure of the liquid over the plane surface at N; p is the equilibrium vapour pressure of the liquid over the concave surface at M and p* is less than P.

The pressure in the liquid just below the meniscus at

$$M = P - h\rho g \quad \ldots \quad \ldots \quad (2)$$

(ρ being the density of the liquid.)

The meniscus may be regarded to be a portion of a sphere of radius r.

There will then be a difference of pressure between its two sides which is equal to $\frac{2T}{r}$.

$$\text{Thus, } p - (P - h\rho g) = \frac{2T}{r} \quad \ldots \quad (3)$$

---

* [If p is not the equilibrium vapour pressure over the curved surface, but less than it, there will be no equilibrium at M and some liquid in the meniscus will evaporate. The amount of water vapour in the enclosure will thereby increase and the level in the tube will fall. But since the conditions are steady, we shall have to presume that the excess of vapour in the enclosure has condensed on the surface at N. It would thereby raise the level of the liquid outside the tube but since this level also is steady, it will have to be further presumed that the amount of water obtained by condensation has risen into the capillary tube to restore the level there to the initial position. The conditions now are again the same as at the beginning and the process therefore repeats itself again. Thus, there will be a continuous circulation of a certain quantity of the liquid going on for any length of time without any expenditure of work by an external agency. But perpetual motion of this type is not found to occur in nature and therefore the assumption that p is not the equilibrium vapour pressure over the concave surface cannot be correct. Therefore p must be the equilibrium vapour pressure; i.e., the saturated vapour pressure over the concave liquid surface at M].

Substituting the value of p in (3) we have

$$(P - h\sigma g) - (P - h\rho g) = \frac{2T}{r}$$

$$\text{or} \quad hg(\rho - \sigma) = \frac{2T}{r}$$

$$\text{or} \quad h = \frac{2T}{r} \cdot \frac{1}{g(\rho - \sigma)}$$

$$\therefore P - p = h\sigma g = \frac{2T}{r} \cdot \frac{\sigma}{(\rho - \sigma)} = \frac{2T}{r} \cdot \frac{\sigma}{\rho}.$$

(since σ is very small as compared to ρ.)

We thus get the interesting results that *the saturated vapour pressure over a concave surface is less than that over a plane surface.*

In the case of liquids like mercury, the meniscus is convex and the liquid column is depressed in the capillary tube. A similar argument as above will show that *the saturated vapour pressure over a convex surface of the liquid is greater than that over a plane surface by an amount equal to* $\frac{2T}{r} \cdot \frac{\sigma}{\rho}$.

The *term saturated vapour pressure used generally refers to the equilibrium pressure over a plane surface.*

182. Formation of drops

Consider a drop of a liquid placed in a space saturated with its vapour (just sufficient for equilibrium with a plane surface). Now that the drop has a convex surface, the existing pressure is less than that required for equilibrium with the drop and the drop therefore begins to evaporate. The rate of evaporation depends upon the excess of pressure, $\frac{2T}{r} \cdot \frac{\sigma}{\rho}$, required for the equilibrium of the drop and it therefore varies inversely as the radius of the drop. *The effect of curvature on the evaporation is small for large drops but becomes appreciable for small drops.* This conclusion has an interesting bearing on the formation of rain drops by precipitation from the water vapour in the atmosphere.

Rain drops cannot grow from a small size to a large size. Consider a small drop of water formed in the atmosphere, say, when

the latter is saturated with water vapour ; the drop requires for equilibrium a vapour pressure several times more than the saturated vapour pressure. It therefore evaporates to produce an increase in the vapour pressure. But as it evaporates its radius diminishes, the vapour pressure required for the equilibrium of the drop becomes still greater and the drop therefore will have no other alternative but to evaporate completely. Thus *rain drops do not actually grow if they have to pass through an initial stage when the size of the drop is very small.*

Thus there is an inherent difficulty for the formation of a cloud in the atmosphere even when the air is saturated. Unless the vapour pressure exceeds the normal value by many times and the vapour is highly saturated, there is no possibility for the vapour to condense into drops.

The *process of condensation, however, is facilitated by the presence of dust particles in the atmosphere.* The dust particles are irregular and they present surfaces which are not very convex. When once condensation starts, it continues, since the radius of curvature increases as the drop grows and the equilibrium vapour pressure becomes less and less.

183. Prof. Aitken has demonstrated the effect of dust particles on the formation of clouds by a simple experiment as follows :

Two vessels A and B are connected by means of a flexible rubber tube R as in Fig. 116. A long glass tube CD, provided with a rubber tube at D and closed by means of a pinch-cock, serves to admit air containing dust into the vessel A when required.

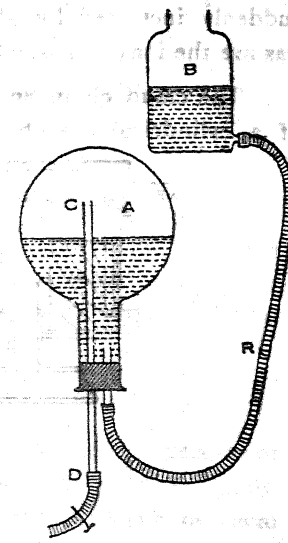


Fig. 116

The vessels A and B are partly filled with water and the space above the water in A contains saturated water vapour. If the vessel B is suddenly lowered, the water from A runs out, causing an immediate expansion of the vapour. The sudden expansion of the

vapour lowers its temperature and therefore results in producing supersaturation in A. A cloud is seen to form in the space above the water in A and it descends slowly and falls into the water below. After the process is repeated a number of times, it is found that the sudden expansion of the vapour fails to produce the cloud. But if now some dust is blown into A through the tube DC and the experiment repeated, the cloud is again seen to form. This proves that dust particles are necessary for the formation of clouds in the air. When the dust particles had been removed into the water below in the experiment by the descent of the drops that were condensed on them there were no nuclei on which the condensation could take place and the cloud did not form. But when dust particles were introduced again, the cloud is formed again.

184. Wilson's cloud chamber

It was for sometime supposed that without dust particles no clouds can be formed. But Prof. C. T. R. Wilson showed that an ionised gas also provides nuclei for the condensation of water vapour. But the degree of supersaturation required for such condensation is rather high and the volume of vapour has to be suddenly increased by about 1·4 times. The nuclei in the ionised gas are the ions (charged atmos or molecules) present in it.

The cloud chamber designed by Prof. C. T. R. Wilson consists of a cylindrical chamber of glass A. It consists of dust-free air

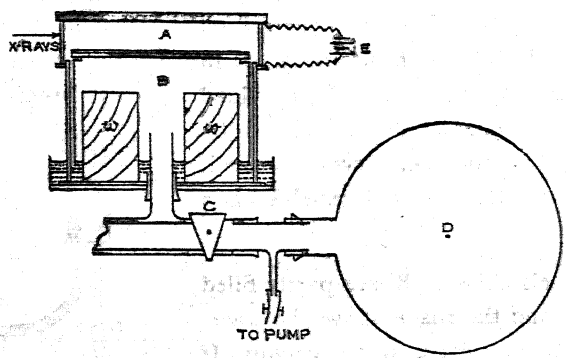


Fig. 117

saturated with water vapour; the water contained at the bottom ensures saturation. Another chamber B is arranged inside A and it

can be jerked down by an automatic device so as to provide for an instantaneous expansion of the vapour in A. The chamber B contains air and is connected to a large vessel D through a valve C. The vessel D is initially exhausted of air. Now, when the valve C is opened, all of a sudden, air from B rushes into D and the chamber B falls down. The extent of fall of B depends upon its initial height. Two wooden blocks *w*, *w* are arranged inside B to reduce the air space in it. *The automatic device for jerking down the chamber B consists in initially evacuating D and suddenly opening the valve C.*

Simultaneous with the opening of the valve C, X-rays which have the property of ionising a gas are admitted into A, through a side window. The sudden expansion of the vapour produce super-saturation and the vapour condenses on the ions produced by the X-rays in the chamber. If the chamber A is immediately illuminated by sending in light from a mercury discharge lamp, the track of the droplets of water (which is also the track of X-rays) can easily be seen. The expansion of the chamber, the admission of X-rays and the illumination of the chamber should be done in quick succession, i.e., almost simultaneously.

If a battery connected between the ceiling and the floor of the chamber A, any stray ions present inside the chamber, to start with, are attracted either towards the ceiling or the floor and removed.

This experiment has been later used to obtain the tracks of charged particles like α and β rays which also ionise the gas along the path pursued by them. A photograph of the chamber reveals a continuous line of small water droplets which is the path of the ionising particle.

185. Detergents

Certain substances produce a change in the properties of liquid surfaces and are therefore known as surface-active agents. Some of them are detergents (or cleansing agents); some of them are wetting agents, and some others are emulsion stabilizers. But detergents include in themselves the cleansing, wetting and emulsifying properties.

Detergents are used to strengthen water in its property of removing dirt from any surface. Dirt which may be of several kinds, is usually held in place by greasy matter. Greases or oils are not soluble in water and therefore water alone cannot remove the grease or dirt. But when a detergent is added to water, it emulsifies the grease and loosens the dirt. The emulsion and the dirt are then rinsed off by water.

Detergents also lower the surface tension of water, when added to it. When the surface tension is lowered, the water penetrates into the small openings on the skin or the fabrics more freely. Cleaning will therefore be more thorough.

Soap is a well-known detergent. But it fails to have appreciable effect when it is to be used with hard water. Synthetic detergents (or Syndets) are nowadays used where soaps fail.

The detergent action of a soap is very interesting. The soap consists of two parts—one, which tries to avoid water but is attracted by oils (or the hydrophobic part) and the other which is attracted by water and not by oils (or the hydrophillic part). The hydrocarbon chain of the soap molecule is hydrophobic while the salt end of the molecule is hydrophillic. When the soap is mixed with water, there is a partial solution occuring at the surface—the hydrophillic end of the molecule entering the water and the hydrophobic end sticking out. But when the soap is put into an oil, the hydrophobic part dissolves in it while the hydrophillic part protrudes out. Thus when soap is used as a detergent, the hydrophobic part dissolves in the grease or oil on the surface to be cleansed but the hydrophillic part projects out and is attracted by water. Thus an oil droplet with a hydrophillic fringe is formed and it can therefore be easily washed away by water.

Examples

1. In an open U-tube mounted on a stand, oil of specific gravity 0·85 and surface tension 28 dynes / cm. is poured. If the radii of cross-section of the tubes are 2·12 mm. and 0·72 mm. respectively, find the difference in levels. (The angle of contact between oil and glass may be assumed to be zero.)

Let h be the difference in the vertical heights of the oil in the limbs.

h is related to the surface tension T by the relation

$$h \rho g = 2 T \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

$$h \times 0.85 \times 980 = 2 \times 28 \left(\frac{1}{0.072} - \frac{1}{0.212} \right)$$

or $$h = \frac{56 \times 0.14}{0.072 \times 0.212 \times 0.85 \times 980} = 0.617 \text{ cm.}$$

2. A capillary tube of glass of internal diameter 0.84 mm. is dipped into a liquid of density 0.84 gm. / c.c. and surface tension 49 dynes / cm. Find the elevation of the liquid into the tube.

Let h be the capillary rise.

Surface tension, T = 40 dynes / cm.

Radius of the tube, r = 0.42 mm. = 0.042 cm.

Density of the liquid = 0.84. gm. / c. c.

$$T = \frac{1}{2} h \rho g r \quad \text{or} \quad h = \frac{2T}{\rho g r}$$

$$\therefore h = \frac{2 \times 49}{0.84 \times 980 \times 0.042} = 2.834 \text{ cm.}$$

3. Calculate the pressure inside a sperical air bubble of radius 0.12 mm. formed below the surface of water (surface tension of water = 72 dynes / cm., atmospheric pressure = 1.013 × 10^6 dynes / sq. cm.)

Surface tension of water = 72 dynes / cm.

Radius of the bubble = 0.12 mm. = 0.012 cm.

Excess of pressure inside a spherical air bubble = $\frac{2T}{r}$

= 2 × 72 / 0.012 = 12,000 dynes / sq. cm.

Total pressure inside = Atmospheric pressure + Excess pressure.

= 1013 × 10^3 + 12 × 10^3 = 1025 × 10^3 dynes / sq. cm.

4. In the drop weight method of determining surface tension, water is allowed to fall slowly down a narrow tube of external diameter 3.8 mm. If 100 drops of water, collected from the tube, weigh 5.334 gm., calculate the surface tension of water.

Radius of the tube, $r = 1{\cdot}9$ mm. $= 0{\cdot}19$ cm.
Mass of each drop, $m = 0{\cdot}05334$ gm.

Surface tension of water, $T = \frac{mg}{3{\cdot}8\, r} = \frac{0{\cdot}05334 \times 980}{3{\cdot}8 \times 0{\cdot}19}$

$= 72{\cdot}4$ dynes / cm.

Exercises

1. Explain the phenomenon of surface tension with suitable illustrations.

2. Define surface tension.

Explain how surface tension is accounted for on kinetic theory.

3. Derive an expression for the excess of pressure inside a cylindrical bubble.

4. Obtain an expression for the excess of pressure inside a spherical soap bubble.

5. Explain what you understand by 'angle of contact'.

Describe how the angle of contact of mercury is determined in the laboratory.

6. Describe with relevant theory how the surface tension of water is determined by the capillary rise method.

7. Describe an experiment to determine the surface tension of water by the method of drops. Derive the formula you use.

8. Describe an experiment to compare the surface tensions of a liquid at different temperatures.

9. Derive an expression for the difference in the saturated pressure of a vapour in contact with a plane surface and that in contact with a curved surface.

10. Write notes on :

(a) spreading of a liquid over another liquid ;

(b) spreading of a liquid over a plane glass plate ;

(c) variation of surface tension of a liquid with temperature ;

(d) formation of clouds ;

(e) Wilson's cloud chamber ;

(f) detergents.

11. A U-tube mounted on a vertical stand contains a liquid of density 0·79 gm. / c.c. and of surface tension 48·5 dynes / cm. If the limbs of the tube are of radii 1·89 mm. and 0·63 mm. respectively, find the difference in vertical heights assuming that the angle of contact of the liquid with glass is zero.

[*Ans.* 1·326 cm.]

12. A capillary glass tube is dipped first into a beaker containing water and then into mercury. Water is found to rise to a height of 2·5 cm. and mercury is found to be depressed to a depth of 0·75 cm. Compare the surface tensions of water and mercury (Angle of contact between water and glass is zero and that between mercury and glass, 130°). [*Ans.* 1 : 6·344]

13. Mercury is poured into a U-tube mounted vertically on a stand. Find the difference in heights of mercury in the two limbs if the tubes are of radii 1·25 mm. and 1·01 mm. (Surface tension of mercury 420 dynes / cm., density of mercury 13·6 gm. / c.c. and angle of contact 135°.) [*Ans.* 0·3124 cm.]

14. Find the pressure inside a soap bubble of radius 0·7 mm., formed in a soap solution if the surface tension of soap solution is 32 dynes / cm. and the atmospheric pressure of mercury is 75·9 cm.

[*Ans.* $1{\cdot}013 \times 10^6$ dynes / sq. cm.]

15. The pressure inside a soap bubble of radius 0·64 mm. is found to balance a column of oil of height 3·2 cm. and of specific gravity 0·8. Find the surface tension of soap solution.

[*Ans.* 40·14 dynes / cm.]

16. In a drop weight determination of the surface tension of an oil, 50 drops are collected down a narrow glass tube of external diameter 4·2 mm. Find the mass of the liquid collected if the surface tension of the liquid is 50 dynes / cm.

[*Ans.* 2·036 gm.]

17. 100 drops of water falling down a tube of external diameter 3·5 mm., is collected under coconut oil of specific gravity 0·8. Calculate the interfacial tension between water and oil if the water collected weighs 12·35 gm. [*Ans.* T = 36·4 dynes / cm.]

18. Explain how liquids exhibit surface tension. Describe, giving the relevant theory, an experiment to find the surface tension of water accurately. [*A. U. March 1961*]

19. State and explain what happens when (i) oil is poured over troubled water and (ii) camphor particles are sprinkled over the surface of water.

Deduce an expression for the excess pressure inside a soap bubble. How is the result verified? [*A. U. Sept., 1962*]

20. Define surface tension of a liquid. Describe the capillary ascent method for the determination of the surface tension of a liquid, giving the derivation of the formula that is used in the determination. How does surface tension vary with temperature? [*A. U. April 1963*]

21. Derive an expression for the excess pressure inside a curved liquid surface. Apply the formula to the case of a cylindrical film and discuss briefly the stability of a cylindrical film. [*A. U. Sept., 1963*]

22. Describe with relevant theory a method of determining he variation of surface tension with temperature.

The difference of pressure between the inside and outside of a soap bubble of diameter 3 mm. is 1 cm. of liquid of specific gravity 0·8. Calculate the surface tension of the soap solution. (Acceleration due to gravity is 980 cm./sec^2.)
(*Ans.* 29·4 dynes/cm.) [*A. U. April 1964*]

23. Describe Quincke's method of finding surface tension. Derive the formula employed. Write a note on the angle of contact. [*M. U. Sept., 1964*]

24. Describe Jaegar's method of finding surface tension deriving the formula employed. What are the limitations of the method? Discuss the conditions leading to the formation of drops. [*M. U. April 1965*]

25. Deduce an expression for the excess of pressure inside a soap bubble in terms of its radius 'a' and surface tension T of the soap solution. Two spherical soap bubbles coalesce. If $\triangle$ V is the

consequent change in volume of the contained air and $\triangle S$ is the corresponding change in total area, show that $3P. \triangle V = 4T. \triangle S$, where P is the atmospheric pressure. [*A. U. April 1965*]

26. Arrive at a relation connecting the excess of pressure inside a curved liquid surface and the principal radii of curvature of the surface.

Write a note on the formation of water drops. [*A. U. March 1965*]

27. Define surface tension and state the units in which it is measured. Describe and outline the theory of one method, by which you will measure the surface tension of a liquid which wets glass. What is the mean pressure inside an air bubble of radius 1 mm., 100 cm., below the surface of a pond? (surface tension of water = 72 dynes/cm., and g = 980 cm./sec^2.)

(*Ans*: $1{\cdot}112 \times 10^6$ dynes / sq. cm.) [*A. U. Sept., 1965*]

28. Explain how the interfacial tension between oil and water may be determined. The difference of pressure between the inside and outside of a soap bubble of diameter 3 mm. corresponds to 0·6 cm. of water. What is the surface tension of soap solution? (g = 980 cm./sec^2.) (*Ans.* 22·05 dynes / cm.)

[*A. U. Sept., 1965*]

29. Find an expression for the vapour pressure for a curved liquid surface. Explain the formation of drops. Write a note on mono-molecular layers. [*M. U. April 1966*]

30. Write notes on surface tension and surface energy.

[*M. U. April 1966*]

31. Explain the terms (a) surface tension, (b) surface energy and (c) angle of contact.

Describe in detail an experimental method for studying the variation of surface tension of water with temperature.

[*A. U. March 1966*]

32. Write short notes on (a) 'Detergents and wetting.'

[*A. U. March 1966*]

(b) Variation of surface tension with temperature.

[*A. U. April 1967*]

33. Derive an expression for the excess pressure inside a spherical drop of liquid. How is the expression modified in the case of a spherical bubble of the liquid?

A drop of water of radius 2 mm. is split into 1,000 tiny drops. Evaluate the pressure inside one of the small drops. (surface tension of water = 70 dynes / cm. (*Ans.* 7,000 + H dynes / sq. cm.)

[*A. U. March 1966*]

34. Explain the terms 'surface tension' and 'surface energy.' Derive an expression for the difference of levels of the liquid inside and outside a capillary tube dipped in a liquid.

When a capillary tube of bore 1 mm. is dipped in mercury, the level of mercury inside is found to be 1 cm. below the level outside. Evaluate the surface tension of mercury. (Angle of contact of mercury = 140°.)

[*Ans.* 435 dynes / cm.] [*A. U. Oct., 1966*]

35. Describe the drop weight method of finding the surface tension of a liquid as well as the interfacial tension between two liquids. Give a simple theory and indicate the refinement needed to give a correct value.

A sphere of water of radius 1 mm. is sprayed into million drops all of the same size. Find the work expended in doing this.

[*Ans.* 12·44 T ergs.] [*A. U. April 1967*]

36. Describe any two experiments to show that the surface of a liquid behaves like a stretched elastic membrane. Describe with necessary theory the capillary rise method of finding the surface tension of a liquid.

[*A. U. Oct., 1967*]

37. Deduce the relation connecting the excess of pressure inside a curved liquid surface and the principal radii of curvature of the surface. Write a note on the formation of water drops.

Chapter X

VISCOSITY

186. A fluid is generally defined as a substance which cannot withstand a shearing stress. All fluids, actually however, do offer resistance when there is relative motion between their different layers and it is necessary to apply tangential forces if the relative motion is to be maintained. *The property by which a liquid or gas offers resistance to relative motion between its different layers is called* **viscosity.**

Since the resistance offered opposes the motion of one layer past another layer and is thus similar to friction, it is also called *internal friction.*

If water, contained in a beaker, is stirred with a rod, it is seen that the water in contact with the rod moves with it, while the water near the walls of the beaker is stationary and the intermediate portions move with different velocities. If the stirring is stopped, the relative motion between the different parts gradually slows down and the water as a whole comes to rest after a short time. The internal forces of viscosity are responsible for bringing the water to rest. It is on account of the viscosity of air, rain drops fall slowly. The viscosity of sea-water makes the waves subside when the wind falls.

187. Coefficient of viscosity

The law relating to viscosity was given by Newton. He assumed that, when the fluid moves in parallel layers, the velocity being different from layer to layer, the relations governing the tangential force F acting on any layer, are as follows:

1. The tangential force is directly proportional to the area of the layer.

2. It is directly proportional to the difference in velocity between two adjacent layers.

3. It is inversely proportional to the distance apart between the layers.

Thus, if A is the area of the layer and v and $v + dv$ are the velocities of two layers separated by a distance dx (in a direction perpendicular to that of v),

$$F \propto A \cdot \frac{dv}{dx} \text{ or } F = \eta \cdot A \cdot \frac{dv}{dx} \quad \ldots \quad \ldots \quad (1)$$

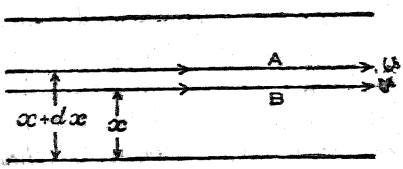


Fig. 118

where η is a constant for the liquid or the gas and is called its **coefficient of viscosity.**

$\frac{dv}{dx}$ is the velocity gradient in the liquid.

Relation (1) is **Newton's Law** of viscous flow in fluids. If in (1), dv = 1 cm./sec.

dx = 1 cm.,

and A = 1 sq. cm.

we get F = η.

Thus, *the coefficient of viscosity of a fluid is the tangential force per unit area required to maintain a relative velocity of unity between two layers which are unit distance apart,*

or, *the coefficient of viscosity of a fluid is the ratio of the tangential force per unit area to the velocity gradient in the fluid.*

The coefficient of viscosity is expressed in dynes per sq. cm., per unit velocity gradient. It differs from liquid to liquid and gas to gas.

188. Dimensions of Viscosity

The coefficient of viscosity η is given by the relation

$$f = -\eta A \cdot \frac{dv}{dx}$$

i.e., $[M L T^{-2}] = \eta [L^2] [L T^{-1} L^{-1}]$.

$\therefore [\eta] = [M L^{-1} T^{-1}]$.

189. Stream-line flow and turbulent flow

When a liquid has to flow through a channel or a tube, it is usualy acted on by an excess of pressure at one end which drives

it forward overcoming the viscous drag on the different layers of the liquid. The velocity of flow, for a given external pressure, depends upon the coefficient of viscosity. If its velocity is less than a certain velocity, called the **critical velocity**, the liquid flows in an orderly manner and the flow is called **steady flow or stream-line flow.** But if the velocity is more than the critical velocity, the orderly flow ceases and its motion becomes **unsteady or turbulent.**

In steady flow (or stream-line flow) the velocity at every point of the stream of the liquid remains constant. The path of any particle in the liquid in stream-line motion will be along a definite line, called **a line of flow**, the direction of which at every point gives the direction of motion of the liquid at that point. Such a line is called a **stream-line.** The lines of flow are parallel to the axis of the tube when the liquid is having stream-line motion.

When the external pressure causing the flow of the liquid is excessive, the motion of the liquid takes place with a velocity greater than the critical velocity. The particles in the liquid then begin to move in curved paths forming eddies or vortices, losing all orderliness. Most of the energy supplied by the external pressure is utilized in the formation of vortices. The flow is then said to be *turbulent* and the liquid is said to be in *turbulent motion.* This may be illustrated by introducing a little colouring matter at a point of the inlet. When the driving pressure is small, the colour spreads in a thin line in the tube indicating stream-line flow. But when the pressure is increased and the velocity of flow exceeds the critical velocity, the orderliness is lost, the motion becomes turbulent and the colour is found to spread across and fill the entire tube.

190. Critical velocity and Reynold's number

When a liquid flows through any tube, the dimensions of the tube also effect the critical velocity. The liquid can have stream-line flow even at higher velocities provided it is viscous enough and the tube is narrow. But in turbulent motion, the density of the liquid is involved since the energy supplied is wasted in the formation of eddies. *High viscosity for any liquid promotes stream-line or steady flow but low viscosity and high density encourage turbulent motion.*

If η is the coefficient of viscosity of a liquid, ρ, its density, and r, the radius of the tube, Osborne Reynolds showed that

$$\text{the critical velocity } V_c = \frac{K\eta}{r.\rho}$$

(where K is a constant called *Reynold's Number*).

For narrow tubes, K was found to be 1,000.

The *rate of flow of a liquid, when it is in steady motion does not depend upon the density of the liquid, but only on its viscosity.*

190. (a) Fugitive Elasticity

From the definition of coefficient of viscosity we have the relation, Tangential viscous stress, i.e., tangential force per unit area = $\eta \times$ relative velocity gradient

$$= \eta \times \frac{du}{dz} = \eta \frac{d}{dz}\left(\frac{dx}{dt}\right)$$

$$= \eta \frac{d}{dt}\left(\frac{dx}{dz}\right) \quad \ldots\ (1)$$

Here u is the velocity in the x direction which is the direction of motion and the z direction, a direction perpendicular to it. The gradient is taken in a direction perpendicular to the direction of motion.

In the case of elastic solids, we have a similar relation for the modulus of rigidity, n, as follows :

$$\text{Tangential shearing stress} = n \times \text{shear, } \theta = n.\frac{dx}{dz} \quad \ldots\ (2)$$

(since the angle of shear θ is equal to the ratio of the relative displacement between parallel layers to the distance between the layers). Here dx is the relative displacement between the layers in the direction of x.

A close similarity exists between relation (1) and (2). This made Poisson and Maxwell consider that *a viscous liquid is the limiting case of an elastic solid, which is breaking down under a shearing stress.* This kind of interpretation is especially apt in the case of substances like pitch and sealing-wax, which behave as solids but break down under a shear and begin to flow (very slowly) even as a fluid. We may thus regard liquids as possessing fugitive elasticity (rigidity)

and if n in relation (2) is the fugitive elasticity of a liquid, we have from (1) and (2)

$$\eta . \frac{d}{dt}\left(\frac{dx}{dz}\right) = n . \frac{dx}{dz} \quad \text{.... (3)}$$

If we write $\frac{dx}{dz} = \theta$, the angle of shear, we will have

$$\eta \ \frac{d}{dt}\left(\theta\right) = n . \theta \quad \text{or} \quad \eta . \frac{(d\theta/dt)}{\theta} = n \quad \text{.... (4)}$$

If we assume that $\frac{d\theta}{dt}$, the rate at which the shear breaks down is proportional to θ, we may write, $\left(\frac{d\theta}{dt}\right) / \theta = \lambda$, a constant.

Relation (4) then becomes $\lambda \eta = n$ or $\eta = \frac{n}{\lambda}$ .... (5)

$\frac{1}{\lambda}$ is called the **time of relaxation.**

Equation (5) gives the relation between the coefficient of viscosity of the liquid and its fugitive elasticity.

191. Determination of η : Poiseuille's method

Poiseuille suggested a convenient method of determining η of a liquid by measuring its rate of flow **through a capillary tube.** It was assumed, in all the experiments on the determination of η by this method, that

(1) the liquid flows in stream-lines parallel to the axis of the tube, and

(2) that there is no slip between the walls of the tube and the liquid, i.e., the layer of the liquid in contact with the walls of the tube is at rest. The velocity of the liquid increases as we approach the axis of the tube and becomes a maximum at the axis.

Let a liquid flow through a capillary tube of radius 'a' under a constant difference of pressure P between its ends.

Consider a cylindrical shell AB of the liquid of radius r and thickness dr.

Let the velocities of the liquid at a distance r and $r + dr$ from the axis be u and $u + du$ respectively.

$\therefore$ The velocity gradient $= -\frac{du}{dr}$.

Hence the viscous drag on the inner cylinder per unit area $= -\eta \frac{du}{dr}$.

(η being the coefficient of viscosity of the liquid.)

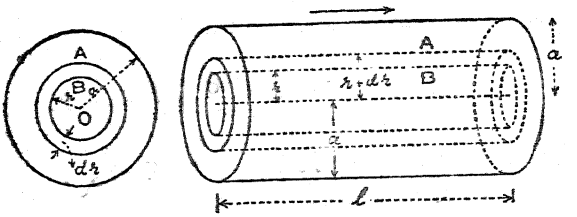


Fig. 119

If l is the length of the tube, the area on which the viscous drag acts $= 2\pi r l$.

$\therefore$ The viscous drag $= -2\pi r l \eta \frac{du}{dr}$ .... (1)

Since the pressure difference between the ends of the tube is P the accelerating force acting on the cylindrical shell $= \pi r^2 P$ ... (2)

When the flow of the liquid is steady, the two forces must be equal so that

$$\pi r^2 P = -2\pi r l \eta \frac{du}{dr} \quad \text{.... (3)}$$

$$\text{or } du = -\frac{P}{2\eta l} r\, dr$$

$$\text{on integration, } u = -\frac{P}{4\eta l} r^2 + C \quad \text{... (4)}$$

(C being the constant of integration.)

But the velocity at the walls of the tube is zero so that when $r = a$, $u = 0$.

$$\therefore 0 = -\frac{P}{4\eta l} \cdot a^2 + C \text{ or } C = \frac{P}{4\eta l} \cdot a^2 \quad \text{... (5)}$$

Substituting for C in (4) we get

$$u = \frac{P}{4\eta l}(a^2 - r^2) \quad \ldots\ (6)$$

This expression gives the distribution of velocity of the liquid in the tube and it can be used to determine η. But it is more convenient to measure the volume of the liquid flowing out per second rather than the velocity of the liquid flowing at a particular distance from the axis.

If dV is the volume of the liquid flowing out per sec. through the cylindrical shell, then

$$dV = 2\pi r.\, dr.\, u$$

$$= \frac{P}{4\eta l}(a^2 - r^2)\, 2\pi r.\, dr \quad \ldots\ (7)$$

$\therefore$ Volume of the liquid flowing out per sec. through the tube

$$V = \frac{\pi P}{2\eta l}\int_0^a (a^2 - r^2)\, r dr$$

$$= \frac{\pi P}{2\eta l}\left[\frac{a^2 \cdot r^2}{2} - \frac{r^4}{4}\right]_0^a$$

$$= \frac{\pi P a^4}{8\eta l} \quad \ldots\ (8)$$

The coefficient of the viscosity of a liquid, η can be determined using this relation.

192. Corrections for the formula

The relation (8) applies strictly in an ideal case when the liquid-flow through the capillary tube is extremely slow so that the liquid just drops down at the outlet end. Otherwise, the above relation requires correction (a) for the kinetic energy acquired by the liquid (utilising a part of the pressure head) and (b) for the acceleration of the liquid at the inlet end of the tube.

(a) Let P_1 be the pressure difference between the ends of the tube which is needed to overcome the viscous drag.

Thus, work done in overcoming viscosity $= P_1 . V$ ergs ... (1)

Volume of the liquid flowing through the shell per sec. $= 2\pi r\, dr\, u$.

Mass of the liquid flowing out per sec. $= 2\pi r.\ dr.\ \rho.\ u.$
(ρ being the density of the liquid.)

$\therefore$ Kinetic energy of the liquid flowing out
$= \frac{1}{2}\ (2\pi r.\ dr.\ \rho.\ u)\ u^2$

$\therefore$ K. E. of the liquid flowing through the entire tube

per sec. $= \int_0^a \pi\, \rho\, u^3\, r\, dr$

But $u = \dfrac{P}{4\eta l}\ (a^2 - r^2)$

$$\therefore \text{K. E. per sec.} = \pi\rho \left(\frac{P}{4\eta l}\right)^3 \int_0^a (a^2 - r^2)^3\, r\, dr$$

$$= \pi\rho \left(\frac{P}{4\eta l}\right)^3 \int_0^a (a^6 - 3a^4r^2 + 3a^2r^4 - r^6)\, r\, dr$$

$$= \pi\rho \left(\frac{P}{4\eta l}\right)^3 \left[\frac{a^6r^2}{2} - \frac{3a^4r^4}{4} + \frac{3a^2r^6}{6} - \frac{r_8}{8}\right]$$

$$= \pi\rho \left(\frac{P}{4\eta l}\right)^3 \frac{a^8}{8}$$

$$= \frac{\rho}{\pi^2 a^4} \left(\frac{\pi P a^4}{8\eta l}\right)^3 = \frac{\rho V^3}{\pi^2 a^4} \quad \ldots \quad (2)$$

If P is the observed difference in pressure between the ends of the tube and V, the volume of the liquid flowing out per sec, total work done = PV. This has been used up in doing work against viscosity and in imparting K. E. to the liquid.

$$\therefore PV = P_1 V + \frac{\rho V^3}{\pi^2 a^4}$$

$$\therefore P_1 = \left(P - \frac{\rho V^2}{\pi^2 a^4}\right).$$

(b) It has been assumed in the stream-line flow that the velocity of the liquid remains the same along the entire length of the tube. But the liquid enters the tube at the inlet end with an acceleration which decreases as it flows along and becomes zero at a certain point C in the tube. The section at C is called **vena contracta**. To correct for this, the length of the tube is supposed to be increased by a certain amount, K. a (K is a constant which is equal to 1·64 in most cases and a is the radius of the tube).

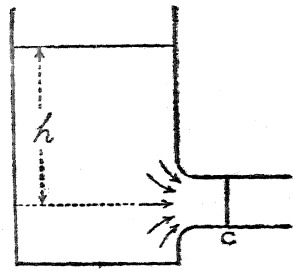


Fig. 120

∴ The corrected length of the tube on this account = $(l + 1{\cdot}64\,a)$
Thus applying the two corrections, the Poiseuille's formula becomes

$$\eta = \frac{\pi \left(P - \frac{V^2 \rho}{\pi^2 a^4} \right) a^4}{8\,V\,(l + 1{\cdot}64\,a)}.$$

η is sometimes called the **dynamic viscosity** while $\frac{\eta}{\rho}$, the **kinematic viscosity.**

193. Experiment: The Poiseuille's formula can be used to determine the coefficient of viscosity of liquids available in large quantities. It is convenient in all these experiments to use a constant pressure head to maintain a constant pressure difference between the two ends of the capilaary tube.

The constant pressure head consists of a cylindrical tank C, provided with a side opening through which a one-holed rubber stopper carrying a capillary tube A is inserted. An inlet tube T brings in water (or liquid) into the tank. An outlet tube B is arranged inside the vessel with its upper end at a convenient height h above the axis of the capillary tube. When the flow of the water from T is properly adjusted, water (or liquid) in the tank is maintained at the same level,

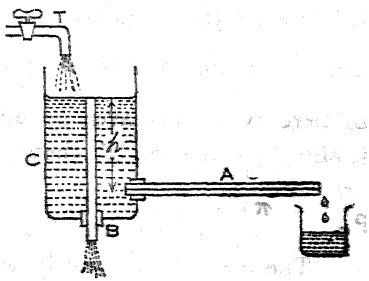


Fig. 121

the excess of liquid flowing out through B. This arrangement ensures a constant difference of pressure between the two ends of the capillary tube, which is equal to $h\rho g$ (ρ being the density of the liquid).

The capillary tube is *first cleaned thoroughly* with acidified potassium dichromate solution and next with tap water. It is then fitted *horizontally* into the rubber stopper in the constant pressure head tank.

Water (or liquid) is then allowed to run down into the tank C through the inlet T, until its level is just above the opening of the outlet tube B. The inflow is then adjusted so that the level remains steady at the height h above the axis of the capillary tube.

A *clean* and *dry* beaker is weighed empty (m_1 gm.). Water flowing out freely from the capillary tube is then collected into the beaker for a known time (say 20 minutes). The beaker is again weighed (m_2 gm.). The mass of water collected in a second is calculated and the volume of water flowing out per sec., V, is obtained by dividing the mass per sec., by the density of water, ρ, at the temperature of the experiment.

The radius of the capillary tube, a, enters in the fourth power in the formula for viscosity. Any slight error in the value of 'a' magnifies the error in the final result to an appreciable extent. To obtain the value for 'a' accurately, the capillary tube is taken out of the tank and dried. It is then filled almost completely with mercury. The length of the mercury thread 's' is measured carefully with a scale. The mass of mercury 'm' is then obtained by emptying it into a weighed crucible and again weighing it. The volume of mercury inside the capillary tube is equal to $\frac{m}{\rho_1}$ (ρ_1 being the density of mercury). The average radius of cross-section of the pellet which is also the value for 'a' above is then calculated from the relation

$$\frac{m}{\rho_1} = \pi a^2 s.$$

The length of the capillary tube 'l' is measured with a scale and the coefficient of viscosity calculated from the relation,

$$\eta = \frac{\pi[P - (V^2\rho/\pi^2 a^4)]a^4}{8V(l + 1{\cdot}64\,a)} = \frac{\pi P a^4}{8Vl} \text{ approximately.}$$

The viscosity of liquid varies with temperature and the temperature of the experiment should therefore be noted.

194. Variation of Viscosity with temperature

The viscosity of a liquid varies with temperature. Its variation with temperature can be studied with the help of an arrangement described below.

A capillary tube AB is immersed in the liquid bath. The end B is joined to a wider bent tube CDE by means of a rubber tube. A bent pin is fastened to that wider tube and its tip is arranged to just touch the liquid surface in the vessel. When the liquid is allowed to siphon out through the tube ABCDE, it will flow out under a pressure head h (= FE) of the liquid. As the liquid flows out of the tube in the experiment, the level in the bath falls. To maintain the same pressure head, h, the bath gently raised so that the tip of the pin is always in touch with the liquid surface in the bath.

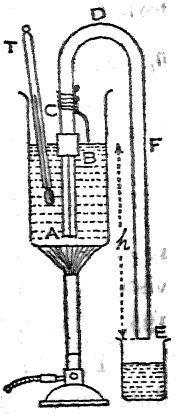


Fig. 122

The liquid flowing out of the end E is collected for a known interval of time and its volume per second calculated. The coefficient of viscosity η can be calculated from the relation $\eta = \frac{\pi P a^4}{8Vl}$, determining '$a$' and '$l$' of the capillary tube as in the above experiment.

To study the variation of viscosity at different temperatures, the experiment is repeated at different temperatures, keeping the temperature steady at least for 4 or 5 minutes at each temperature. The viscosity of a liquid is found to decrease with temperature and a graph between viscosity and temperature will be as in Fig. 123.

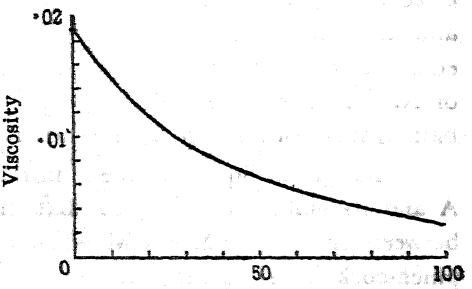


Fig. 123

If η_1 and η_2 are the coefficients of viscosities at temperatures t_1°C and t_2°C

then $\eta_1 \propto \frac{P_1}{V_1}$ and $\eta_2 \propto \frac{P_2}{V_2}$

or $\frac{\eta_1}{\eta_2} = \frac{P_1 V_2}{P_2 V_1}$.

However there is no relation which expresses the variation of viscosity with temperature accurately. *Slotte* suggested an empirical formula $\eta_t = \frac{\eta_o}{(1 + \alpha t + \beta t^2)}$ where α and β are constants and η_t, η_o are the viscosities at t°C and 0°C respectively.

Another formula based on certain assumptions was suggested. If η_T is the coefficient of viscosity at any temperature T°A then

$\eta_T v^{\frac{1}{3}} = A. e^{\frac{c}{vT}}$ where A and c are constants and v the specific volume of the liquid. This expresses the relation between viscosity and temperature fairly accurately.

195. Ostwald's Method

The viscosities of different liquids or of the same liquid at different temperatures can be compared in a more simple manner using Ostwald's apparatus.

Ostwald's apparatus consists of a U-tube ABCDE with a capillary section as in the Fig. 124. Marks K, L, M and N are taken on the tube as shown and the end E is provided with a rubber tube fitted with a pinch-cock, T. The bulbs are nearly equal in volume and the marks are such that the volumes between K and L is equal to that between M and N. The level of L is above that of N. This arrangement may be immersed in a suitable liquid bath, if it is desired to study the variation of viscosity with temperature.

The given liquid is then introduced into the tube through A and its volume adjusted so that the liquid occupies the portion between the marks K and M when the pinch-cock is closed. The pinch-cock T is opened, air sucked out of the rubber tube so that the level of the liquid rises up above the mark K, and then the pinch-cock is closed. On opening the pinch-cock again, the liquid begins to flow down the tube EC into the tube CA. The time

taken by the liquid to fall from the mark K to the mark L is noted. The experiment is repeated three or four times and by average, time t_1 is noted.

The experiment is then repeated with the second liquid also and the mean time t_2 taken by it to fall from K to L obtained.

If V is the volume of the liquid which flows through the capillary tube as the level falls from K to L, the volumes V_1 and V_2 of the two liquids flowing through the capillary tube in a second are equal to $\frac{V}{t_1}$ and $\frac{V}{t_2}$ respectively.

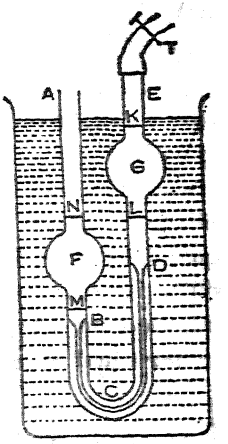


Fig. 124

$$\therefore V_1 = \frac{V}{t_1} \text{ and } V_2 = \frac{V}{t_2}.$$

The coefficient of viscosity of a liquid varies inversely as the fourth power of the radius of the tube. The wider portions of the tube may therefore be supposed to have no effect on the rate of flow of the liquid and that the capillary portion alone determines the rate of flow. If η_1 and η_2 are the coefficient of viscosities of the two liquids respectively,

$$\eta_1 = \frac{\pi P_1 a^4}{8 V_1 l} \text{ and } \eta_2 = \frac{\pi P_2 a^4}{8 V_2 l}.$$

The pressures P_1 and P_2 go on changing as the liquid flows out of the capillary tube. But for any position of the level of the liquid in the tube EC, the pressure is proportional to the density of the liquid. Hence for every corresponding positions of the liquids,

$$P_1 \propto d_1 \text{ and } P_2 \propto d_2,$$

d_1 and d_2 being the densities of the liquids.

$$\therefore \frac{\eta_1}{\eta_2} = \frac{\pi d_1 a^4}{8 (V/t_1) l} \Big/ \frac{\pi d_2 a^4}{8 (V/t_2) l}$$

$$= \frac{d_1 t_1}{d_2 t_2}.$$

Thus knowing the densities of the liquids and measuring the times taken by them respectively, the viscosities can be compared. The viscosities of a liquid at different temperatures can be compared

in the same manner by determining the times in the Ostwald's bulb and knowing the corresponding densities at different temperatures.

196. Viscosity of highly viscous liquids: Stokes's method

Highly viscous liquids do not easily flow through capillary tubes and Poiseuille's method of determining η, therefore fails. Sir George Stokes suggested that in such cases, η can be determined by studying the motion of a small metallic sphere, down the liquid.

Any metallic sphere dropped into a liquid begins to move down with acceleration under gravitational pull. But its motion in the liquid is opposed by viscous forces in the liquid which also increases in magnitude as the velocity of the sphere increases. Finally a velocity will be attained when the apparent weight of the body becomes equal to the retarding viscous forces acting on it. Consequently there will be no further acceleration and the sphere continues to move down with the same velocity thereafter. This uniform velocity is also called the **terminal velocity.**

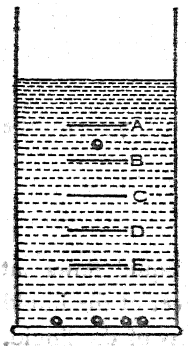


Fig. 125

Stokes obtained a mathematical relation, on Hydrodynamical considerations, for the viscous drag on such a sphere moving down in an infinite medium with a velocity v. The viscous drag, F according to Stokes is given by

$$F = 6 \pi a \eta v \quad \ldots \quad (1)$$

where η is the coefficient of viscosity of the liquid.

Let a be the radius of the sphere,

ρ, the density of the sphere

and σ the density of the liquid.

$\therefore$ The apparent weight of the sphere $= \frac{4}{3}\pi a^3 (\rho - \sigma) g \quad \ldots \quad (2)$

If v is the terminal velocity,

$$6\pi a \eta v = \frac{4}{3} \pi a^3 (\rho - \sigma) g$$

$$\text{or} \quad \eta = \frac{2}{9} \frac{a^2}{v} (\rho - \sigma) g \quad \ldots \quad (3)$$

But in practice, it is not possible to have an infinite medium for the sphere to move down. For accurate results, corrections will have to be applied for the finite size of the container.

Experiment: The given liquid is taken into a tall jar of fairly large diameter. Four or five marks A, B, C, D ... are drawn on the

jar at equal distances to indicate the levels so that the velocity of the falling sphere may easily be calculated with reference to them. A steel ball (from ball bearings) is gently dropped centrally into the jar. The times taken by the ball to move down the lengths AB, BC and CD are noted. (If they are equal, the ball is moving down with the terminal velocity. Otherwise, a ball of smaller size should be selected so as to obtain this condition). The terminal velocity v is calculated from the observations.

The mean diameter of the ball ($2a$) is accurately measured with a screw gauge. The coefficient of viscosity η is then calculated from the relation, $\eta = \frac{2}{9}\frac{a^2}{v}(\rho - \sigma)g$.

(ρ is the density of steel and σ, the density of the liquid).

***196. (a) Explanation of viscosity of a gas on kinetic theory**

An explanation for the viscosity of a gas was given by Maxwell on the basis of kinetic theory.

Let us consider a gas flowing steadily in the direction of the arrows (Fig. 126). Consider a plane MN in the gas and let the velocity of flow of the gas be v along MN.

Let AB and CD be two planes parallel to MN, which are respectively above and below MN and let their distances from MN be each equal to λ, the mean free path of the molecules of the gas. Let $(v + dv)$ and $(v - dv)$ be the velocities of flow of the gas along AB and CD respectively.

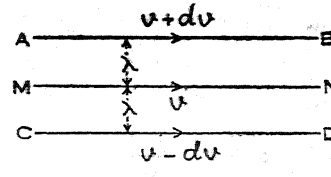


Fig. 126

According to the kinetic theory, the molecules of the gas have random movements and as a consequence of these random movements, molecules of the gas will be moving across MN from the layers above and below MN. Let n be the number of molecules per c.c. of the gas. We may, for simplicity, assume that these n molecules are divided into three equal groups, one group moving perpendicular to the plane MN and the other two groups moving in perpendicular directions, parallel to the plane MN. Of the $(\frac{1}{3}n)$ molecules moving perpendicular to MN, one half may be supposed to

**Omitted for Andhra University.*

be moving downwards and the other half upwards, at any instant. Thus it may be assumed that the number of molecules per c.c. in the gas which are moving perpendicular to MN in the downward direction is equal to $\frac{n}{6}$. If $\overline{c}$ is the average velocity of the molecules of the gas, the number that passes unit area of MN in the downward direction in 1 sec. $= \frac{n\,\overline{c}}{6}$. Similarly the number that passes unit area of MN in the upward direction in 1 sec. is also equal to $\frac{n\,\overline{c}}{6}$. These two sets of molecules encounter in the plane MN and we may consider the result of these encounters.

The molecules passing MN in the downward direction would have had their collisions for the last time at a distance, λ above MN and therefore they would belong to the layer AB. Consequently all these $\frac{n\,\overline{c}}{6}$ molecules which reach unit area of MN in 1 sec. from AB, would have the velocity $(v + dv)$ in addition to their random velocities.

If m is the mass of a molecule, the momentum in the direction of flow of the gas possessed by a molecule, $= m\,(v + dv)$.

Therefore the molecule, which pass unit area of MN in the downward direction carry with them a directed momentum $= m\,(v + dv)\,\frac{n\,\overline{c}}{6}$.

Similarly the molecules which pass unit area of MN in the upward direction come from the layer CD which have a velocity equal to $(v - dv)$ and they carry a directed momentum $= m\,(v - dv)\,\frac{n\,\overline{c}}{6}$ in the upward direction.

Since the flow is steady, the velocity of the layer MN is now altered and consequently there is a net transfer of momentum in the downward direction which is equal to

$$\frac{m\,n\,\overline{c}}{6}\,(v + dv) - \frac{m\,n\,\overline{c}}{6}\,(v - dv) = \frac{m\,n\,\overline{c}}{3}\,.\,dv$$

per unit area per sec.

Due to this transfer, the layer above tends to be retarded and the layer below accelerated. The net momentum transferred across unit area in unit time measures the viscous force f, which according to the definition is given by $f = \eta \frac{dv}{dx} = \eta . \frac{dv}{\lambda}$

$$\text{i.e., } \eta . \frac{dv}{\lambda} = \frac{m n \bar{c}}{3} . dv \quad \text{or} \quad \eta = \frac{m n \bar{c} \lambda}{3} \qquad \ldots\ (1)$$

$$= \frac{\rho . \bar{c} \lambda}{3} \qquad \ldots\ (2)$$

Since the mean free path, $\lambda = \frac{1}{\pi \sigma^2 n}$ (vide Art. 133)

equation (1) may be written as $\eta = \frac{m \bar{c}}{3 \pi \sigma^2}$ — (3)

This equation does not contain the number of molecules per c.c. of the gas. It therefore follows that *the viscosity of a gas is independent of its density.* But the density of a gas is directly proportional to the pressure of the gas when its temperature is constant. Application of the kinetic theory to the study of the viscosity of gases therefore leads to the conclusion that *the viscosity of a gas is independent of the pressure of the gas.* This conclusion is found to be in agreement with the results of experiments conducted on gases, except when the pressures are low, and the fact is taken to be significant evidence in support of the kinetic theory of gases.

From equation (3) it is seen that $\eta \propto \bar{c}$, i.e., $\eta \propto \sqrt{T}$.

The coefficient of viscosity should be proportional to the square root of the absolute temperature. Though the viscosity of a gas is found to increase with temperature, the theoretical relation $\eta \propto \sqrt{T}$ is not strictly obeyed.

197. Friction and lubrication

A body placed on a horizontal plane and acted on by a small horizontal force f applied parallel to the plane, does not in general move. This is due to the fact, that an equal force but opposite to f comes into action at the surface of separation. This opposing force is called **friction.** As the applied force is increased gradually, the

body begins to move at a certain stage. The force F which sets the body in motion is called the **limiting friction.** Experiments reveal that the limiting friction bears a constant ratio to the normal reaction R (which is equal and opposite to the weight W of the body when it is at rest). The constant is known as the **coefficient of static friction** and it is denoted by μ.

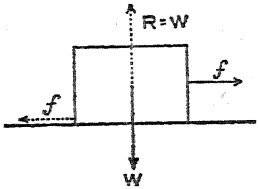


Fig. 127

$$\therefore \mu = \frac{F}{R} \quad \text{or} \quad F = \mu R.$$

To keep a body in continuous motion, a force should constantly be acting on it to overcome friction. This force needed to keep it on in uniform motion is found to be less than that required to set the body in motion and is called **sliding friction.** Consequently the coefficient of sliding friction is less than the coefficient of static friction. The coefficient of friction is different for different pairs of substances.

Friction is sometimes purposely introduced by way of friction clutches to serve as brakes to a running machinery. It is also used to transmit power from a motor to a machine through a belt. The nuts, nails etc., hold on to their positions or, the men and animals are able to walk, due to the friction between the wall and the nail or the foot and the ground respectively.

Dr. Bowden explains that friction is developed because of the unevenness of the contact surfaces. However smooth the surfaces may appear to be, there will certainly be a number of elevations and depressions, the height or the depth of which are of the dimesions of a molecule. When a body moves over another surface, the elevations in one of the contact surfaces get interlocked in the depressions of the other. Consequently a force is required to dislodge these elevations on the moving body. This explains the need for a force to overcome the interlocking effect.

Friction is a dissipating force and it should be minimised for increasing the efficiency of a machine. For reducing friction, a drop or two of some oil (or dust of some soft material) is put between the contact surfaces. The oil (or the material) is called a **lubricant** and the process of reducing friction, '**lubrication**'.

The oil (or the fine dust) enters into the depressions in the contact surfaces and also forms a thin layer so that direct contact between the surfaces is prevented. When the solid surfaces now move, it will be the viscous forces in the lubricant in between that offer the resistance and this fluid friction is always less than the friction between two solid surfaces.

However efficient a lubricant may be, it only reduces friction but cannot altogether eliminate it. It was found that an effective lubricant should have the following properties :

1. It should be sufficiently oily to spread between the surfaces and keep them separate under the working conditions of pressure and temperature.

2. It should be chemically inert and should not undergo any decomposition at high temperatures.

3. I should remain in the same state and should not get squeezed out.

4. It should have low solidifying power if it is a liquid.

5. It should be capable of conducting away heat produced by friction, as soon as it is produced.

Prof. Osborne Reynold showed that the force F required to maintain a uniform relative motion, v, between two surfaces where a lubricant is present is given by $F = \eta \frac{av}{t}$,

η being the coefficient of viscosity of the lubricant and t, the thickness of the film and a, the area of each of the contact sufaces. Generally, for high speeds and low pressures, highly viscous oils such as mobil oil, grease etc., are used; and for low speeds, oils of lower viscosity are used. It may be interesting to note that air also acts as a sort of lubricant in reducing friction. The friction between two metallic surfaces in vacuum is found to be greater than that in air.

Examples

1. *Calculate the coefficient of viscosity of water from the following data obtained in the capillary flow method :*

Length of the capillary tube, l = 36·3 cm. ;

radius of the capillary bore, a = 0·64 mm. ;

height of the liquid above the capillary tube, h = 40 cm.

and rate of flow of water, v = 0·5738 c.c. / sec.

$$\therefore \eta = \frac{\pi h \rho g a^4}{8 v l} = \frac{\pi \times 40 \times 1 \times 980 \times (0.064)^4}{8 \times 0.5738 \times 36.3}$$

$$= 0.0124 \text{ c. g. s. units.}$$

2. *Calculate the root mean square velocity of oxygen given that its density is 1·43 gm. / litre at N. T. P. Determine also its molecular free path and the coefficient of viscosity given that the number of molecules per c.c. = 2.79×10^{19} at N. T. P. and the molecular diameter = 2.3×10^{-8} cm.*

$$\sqrt{\overline{c^2}} = \sqrt{3P/\rho}$$

$$= \sqrt{(3 \times 76 \times 13.6 \times 980) / 0.00143}$$

$$= 4.61 \times 10^4 \text{ cm./sec.}$$

$$\text{Mean free path, } \lambda = \frac{1}{\pi \sigma^2 n}$$

$$= \frac{1}{\pi \times 2.3^2 \times 10^{-16} \times 2.79 \times 10^{19}}$$

$$= 2.16 \times 10^{-5} \text{ cm.}$$

$$\text{Coefficient of viscosity} = \tfrac{1}{3} \rho c \lambda$$

$$= \tfrac{1}{3} \times 1.43 \times 10^{-3} \times 4.61 \times 10^4 \times 2.162 \times 10^{-5}$$

$$= 4.751 \times 10^{-4} \text{ c. g. s. units.}$$

Exercises

1. Describe an experiment to determine the coefficient of viscosity of a liquid by Poiseuille's method.
   Derive the formula you use.

2. Define 'coefficient of viscosity.'
   What are its dimensions?
   Describe, with relevant theory, how the coefficients of viscosity of liquids are compared.

3. Explain what you understand by viscosity.
   Describe with necessary theory how the viscosity of a highly viscous liquid is determined.

4. Write notes on:

(a) Stream-line flow and turbulent flow.

(b) Lubrication.

(c) Critical velocity.

5. Calculate the coefficient of viscosity of hydrogen from the following data:

$\rho = 0.089$ gm. / litre; $\bar{c} = 1.84 \times 10^5$ cm. / sec.;

$\lambda = 1.57 \times 10^{-5}$ cm. [*Ans.* 0.86×10^{-4} c. g. s. units.]

6. A capillary tube is connected to a cistern very near to its bottom. It is first filled completely with water of sp. gr. 1 and coefficient of viscosity 0·01 units and then with a liquid of sp. gr. 0·8 and coefficient of viscosity 0·02 units. Compare the rates of flow. [*Ans.* 1 : 0·4]

7. A cylindrical glass vessel held vertically is fitted with a capillary glass tube of length 40 cm. and diameter 0·6 mm. at a depth of 50 cm. below its open end. If the vessel is completely filled with water, find the rate of flow of the liquid, if the coefficient of viscosity of water is 0·0125 c. g. s. units. [*Ans.* 0·03116 c. c. per sec.]

8. Calculate the mean free path of air given that its density is 1·25 gm· / litre at N. T. P. and its coefficient of viscosity is 1.7×10^{-4} c. g. s. units. [*Ans.* 8.28×10^{-6} cm.]

9. Obtain an expression for the rate of flow of a viscous liquid through a capillary tube. Describe an experiment to compare the coefficient of viscosity of two liquids. [*A. U. Sept., 1961*]

10. What do you mean by (a) stream-line flow, (b) fugitive elasticity and (c) Reynold's number. Deduce Poiseuille's equation for the flow of a liquid through a narrow tube. [*A. U. April 1962*]

11. Describe Poiseuille's formula for the quantity of a viscous liquid flowing through a capilllary tube. Indicate the corrections that are to be applied to the formula. Describe an experiment for the determination of viscosity of a liquid making use of the above formula. [*A. U. April 1963*]

12. Describe with necessary theory an experiment for comparing the viscosities of a liquid at two different temperatures. Discuss

the variation of viscosity of a liquid with temperature and pressure. Write a note on lubrication. (*A. U. Sept., 1963*)

13. Define coefficient of viscosity and find its dimensions. Derive Poiseuille's equation for the rate of flow of a viscous liquid through a horizontal capillary tube. Describe an experiment to compare the coefficient of viscosity of two given liquids.
(*A. U. Sept., 1964*)

14. How will you determine the coefficient of viscosity of water? Derive the formula you use. (*A. U. April 1965*)

15. Obtain Poiseuille's formula for the quantity of a viscous liquid flowing through a capillary tube in a given time. How is the formula employed to compare the coefficients of viscosity of two liquids? (*A. U. April 1965*)

16. What do you understand by the dimensions of a physical quantity? Define the coefficient of viscosity of a fluid and use your definition to find the dimensions of the coefficient.

Water flows first through a horizontal tube of 1·2 mm. bore diameter and then through a horizontal tube of 1 mm. bore diameter joined to the end of the first. If each tube is 10 cm. long and the total pressure drop across the two tubes is 8 cm. of water, what is the pressure drop across the wider tube? (*A. U. Sept., 1965*)
[*Ans.* 2·603 cm. of water.]

17. Describe with theory, Stokes's method of determining the viscosity of a highly viscous fluid.
Write a note on lubricants. (*M. U. Sept., 1963*)

18. Deduce an expression for the volume of a viscous fluid flowing through a capillary tube in unit time.
What method would you employ to find the viscosity of a highly viscous fluid? (*M. U. April 1964*)

19. Write notes on :
(a) Liquid flow and critical velocity. (*M. U. Sept., 1964*)
(b) Stokes's formula. (*M. U. April 1965*)
(c) Turbulent motion. (*A. U. Oct., 1967*)

20. How can the viscosity of a highly viscous fluid be determined? Deduce the formula employed. Write a note on lubrication. (*M. U. April 1966*)

21. Define 'coefficient of viscosity.' Describe with the relevant theory a suitable method of determining the viscosity of water. (*A. U. March 1966*)

22. Deduce an expression for the volume of a viscous fluid flowing through a capillary tube.

Describe a method of finding the viscosity of a highly viscous liquid. (*A. U. April 1967*)

23. Define the coefficient of viscosity. Describe a method of comparing (i) viscosities of two liquids and (ii) the radii of two capillary tubes. Write a note on 'viscosity and lubrication.' (*A. U. Oct., 1967*)

24. Define the coefficient of viscosity and deduce its dimensions.

Derive Poiseuille's equation for the rate of flow of a viscous liquid through a capillary tube.

Describe an experiment to compare the coefficient of viscosity of two liquids. (*A. U. April 1968*)

25. Write a note on lubrication. (*A. U. Oct., 1968*)

Chapter XI

HYDROSTATICS

A. THRUST AND PRESSURE

198. Hydrostatics deals with the study of forces acting inside a fluid when it is at rest. The starting point for the study is the definition of the fluid itself: *A fluid is a portion of matter in a state in which it is incapable of resisting permanently a shearing stress.*

199. Tangential and normal forces

Forces applied parallel to the surface of a body are called *tangential forces* and those applied perpendicular or normal to the surface are called *normal forces.*

Tangential forces tend to alter the shape of the body by causing a relative displacement of the layers in it. If they act on a fluid, the fluid will be caused to flow however small the tangential forces may be (when they act for a sufficient length of time). *When a fluid is at rest, therefore, it is to be inferred that no tangential forces are acting on it and that it is in equilibrium under the action of normal forces only.* Thus if CD be a plane that divides a liquid at rest into two portions A and B, the force f_1 exerted by A on B and the force f_2 exerted by B on A are each normal to CD. Since the liquid is in equilibrium, it also follows that the two normal forces are equal and opposite, constituting a normal stress across CD.

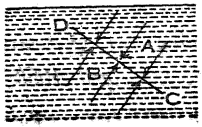


Fig. 128

200. Thrust and pressure

A normal stress acting on a body may tend to compress it or dilate it. It is called a *thrust* when it tends to compress the body and a *tension* when it tends to increase the volume of the body.

The thrust per unit area of the surface is called *pressure.* If a thrust F acts over an area A, then the pressure $P = \frac{F}{A}$, when the thrust is uniformly distributed over the area.

If the thrust is not uniformly distributed over the surface, the pressure P at a point is given by the ratio of the normal force, δf, acting on a small area, δa, when δa is made infinitely small.

$\therefore$ The pressure at a point, P =

$$\lim_{\delta a \to 0} \frac{\delta f}{\delta a} = \frac{df}{da}.$$

201. Proposition I: *Pressure at a point inside a fluid at rest is the same in all directions.*

Consider a small triangular prism ABCC'B'A' in the interior of a fluid with sides AB and A'B' vertical and BC and B'C' horizontal so that $\angle ABC = \angle A'B'C' = 90°$ (Fig. 129).

Let the prism be so small that the different parts of it may be supposed to be at the same depth below the surface of the liquid and the pressures on the faces are uniform.

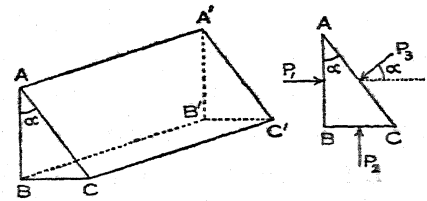


Fig. 129

Let P_1, P_2 and P_3 be the pressures respectively on the faces AA'B'B, BB'C'C and AA'C'C. Let AB = c, BC = a and AC = b;

AA' = BB' = CC' = l and $\angle BAC = \angle B'A'C' = \alpha$.

The different forces acting on the prism are:

(1) the thrust on the face AA'B'B = $P_1 \times c \times l$

(2) the thrust on the face BB'C'C = $P_2 \times a \times l$

(3) the thrust on the face AA'C'C = $P_3 \times b \times l$

(4) the weight of the fluid in the prism = $\frac{1}{2} a.c.l.\rho.g$ (ρ being the density of the fluid)

and (5) the horizontal thrusts on the end faces ABC and A'B'C'.

Now these two thrusts in (5) are acting in directions which are perpendicular to those along which the thrusts in (1) to (4) are acting. Hence they are in equilibrium between themselves and do not come into consideration when the equilibrium of those in (1) to (4) is discussed.

Now resolving the thrust parallel to BC,

$P_1 . c . l = P_3 . b \cos \alpha . l$

or since $c = b \cos \alpha$, $P_1 = P_3$ ... (1)

Resolving perpendicular to BC,

$P_2 . a . l = P_3 . b \sin \alpha . l + \frac{1}{2} a c l \rho g$.

Since the prism is very small and reduces to a point in the limit, the weight of the fluid in the prism becomes zero in the limit.

Hence $P_2 \cdot a \cdot l = P_3 \cdot b \sin \alpha . l$

or $P_2 = P_3$ ... (2)

From (1) and (2). $P_1 = P_2 = P_3$.

Now, since the prism can be taken in any direction in the liquid and α can have any value, P_3 can be made to have any direction we please, so that it may be stated thus: *The pressure at a point in a fluid at rest is the same in all directions.*

202. Proposition II: *The pressure in a liquid at rest is the same at all points at a horizontal plane.*

Let M and N be two points in a horizontal plane inside a homogeneous fluid at rest.

Join MN and construct a thin cylinder with MN as axis and having a sectional area '*a*', which is very small.

Let P_1 and P_2 be the pressures at M and N respectively.

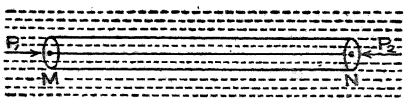


Fig. 130

The forces acting on the cylinder of fluid are:

(1) its weight acting vertically downwards,

(2) the thrusts on the curved surface of the cylinder which are perpendicular to the surface at every point and are therefore normal to the axis MN,

(3) the thrusts, $P_1 \cdot a$, and $P_2 \cdot a$, acting horizontally on the end faces at M and N respectively.

Since the cylinder of fluid is in equilibrium, the forces acting on it in the horizontal direction must themselves be in equilibrium,

i.e., $P_1 \cdot a = P_2 \cdot a$

or $P_1 = P_2$.

Since the relation holds good for any two points in a horizontal plane, it holds good for all the points in the plane.

Hence *the pressure at any point in a horizontal plane in a fluid at rest is the same.*

203. Proposition III: *Difference of pressure between two points in a fluid at rest* $= h \rho g$.

Consider two points M and N on a vertical line inside a homogeneous fluid.

Imagine that a thin cylinder of area of cross-section 'a' is constructed round MN itself as axis.

Let h be the height of the cylinder,

P_1 the downward pressure on the face M,

P_2 the upward pressure on the face N

and ρ the density of the liquid.

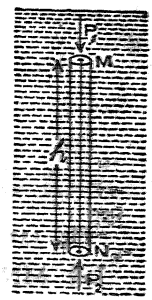


Fig. 131

The fluid cylinder MN is in equilibrium under the action of the following forces:

(1) the weight of the liquid which acts downwards ($= ah\rho g$)

(2) the downward thrust on the face M ($= P_1 \cdot a$)

(3) the upward thrust on the face N ($= P_2 \cdot a$) and

(4) the thrusts on the curved surface of the cylinder which are normal to the vertical axis MN and are therefore horizontal.

Resolving vertically, $P_1 a + h \rho g a = P_2 a$.

or $(P_2 - P_1) = h \rho g$.

Hence *the difference of pressure between two points in a liquid at rest varies at the vertical height between them.*

203. (a) The two points M and N need not be in the same vertical line. The difference in pressure between them is still proportional to the difference in their vertical heights, M and N being points just below the meniscus in the two limbs as in Fig. 132.

Let AB and CD be horizontal and AM, BC and DN vertical.

Pressure at A = Pressure at M + (AM) ρg ... (1)

Pressure at B = Pressure at C + (BC) ρg ... (2)

Pressure at C = Pressure at D
= Pressure at N + (DN) ρg ... (3)

From (1), (2) and (3)
Pressure at M + (AM) ρg = Pressure at N + (BC + DN) ρg

or

Pressure at N = Pressure at M + (AM − BC − DN) ρg
= Pressure at M + $h \rho g$.

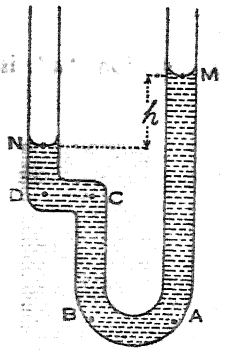


Fig. 132

Now if M and N are at the same level in the fluid but are not connected and the line MN does not wholly lie in the fluid, the same argument can be applied to them and since $h = 0$,

pressure at M = pressure at N.

It follows that when the point M is in the surface, the pressure at N, taken at a depth h below the surface, is equal to $h \rho g$.

204. Proposition IV: *The surface of a liquid at rest is horizontal.*

Let MN be the surface of a liquid exposed to a uniform pressure so that

the pressure at M = the pressure at N ... (1)

If MN is not horizontal, let MN′ be drawn horizontal through M so that N′ is a point vertically below N.

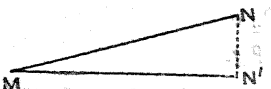


Fig. 133

Since M and N′ are, by construction, in the same horizontal plane,

the pressure at M = the pressure at N′ ... (2)

∴ From (1) and (2)
the pressure at N = the pressure at N′
or the pressure at N − the pressure at N′ = 0.

If h be the height between N and N′,

the pressure at N′ − the pressure at N = $h\rho g$

∴ $h\rho g = 0$. i.e., $h = 0$ (∵ ρ and g are not equal to zero),

Hence the vertical distance between N and N′ should be zero or, N coincides with N′.

Hence *the free surface of a liquid at rest is horizontal.*

205. Proposition V: *The interface of two immiscible liquids at rest is horizontal.*

Let A and B be two liquids of densities ρ_1 and ρ_2 contained in a vessel, (ρ_1 being less than ρ_2).

Let M and N be two points in a horizontal place inside the lower liquid of density ρ_2.

∴ The pressure at M = the pressure at N.

Draw vertical line KM and LN from M and N respectively to meet the surface of the upper liquid in K and L.

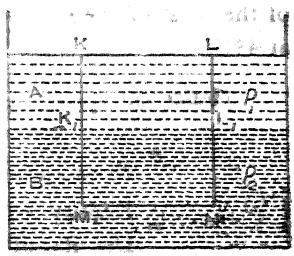


Fig. 134

Let the two lines meet the surface of separation between the two liquids at K_1 and L_1.

$$\begin{aligned}
\text{Now, pressure at M} &= KK_1 \cdot \rho_1 \cdot g + K_1M \cdot \rho_2 \cdot g \\
&= [KK_1 \cdot \rho_1 + (KM - KK_1)\rho_2]\, g \\
&= [KK_1 (\rho_1 - \rho_2) g + KM \cdot \rho_2 \cdot g] \\
\text{Pressure at N} &= LL_1 \cdot \rho_1 \cdot g + L_1N \cdot \rho_2 \cdot g \\
&= [LL_1 \cdot \rho_1 + (LN - LL_1) \cdot \rho_2]\, g \\
&= [LL_1 (\rho_1 - \rho_2) g + LN \cdot \rho_2 \cdot g].
\end{aligned}$$

But the pressure at M = the pressure at N.

$$\therefore KK_1 (\rho_1 - \rho_2) g + KM \cdot \rho_2 \cdot g = LL_1 (\rho_1 - \rho_2) g + LN \cdot \rho_2 g$$

Since M and N are on the same horizontal line and since the surface of the liquid at rest is horizontal KM = LN.

$$\therefore KK_1 (\rho_1 - \rho_2) g = LL_1 (\rho_1 - \rho_2) g$$

or $KK_1 = LL_1$.

Hence K_1 and L_1 are at the same depth below the free surface of the upper liquid or *the interface of the liquids is also parallel to KL and MN and therefore horizontal.*

B. DENSITY AND SPECIFIC GRAVITY

206. The density of a substance is defined as the mass per unit volume of the substance. It is expressed in gm. / c.c. in the C. G. S. system of units and in lb. / c. ft. in the F. P. S system of units.

The density of water is maximum at 4°C and is equal to 1 gm./c.c. or 62·4 lb. / c.ft.

207. The specific gravity of a substance is defined as the ratio of the mass of the substance to the mass of an equal volume of water at 4°C.

Hence the specific gravity of a substance

$$= \frac{\text{Mass of the substance}}{\text{Mass of an equal volume of water at } 4°\text{C}}$$

$$= \frac{\text{Density of the substance.}}{\text{Density of water at } 4°\text{C.}}$$

Thus the specific gravity of a substance can also be defined as the density of the substance to the density of water at 4°C.

Specific gravity of a substance is expressed as a mere number ince it is the ratio of two masses and it will have the same value in the two systems of units.

The specific gravity of a substance in the C. G. S. system is equal to its density of water at 4°C is 1 gm./c.c. But in the F. P. S. system, the density of a substance is equal to 62·4 times its specific gravity since the density of water at 4°C is equal to 62·4 lbs.

208. Specific gravity of a mixture

To find the specific gravity of a mixture:

(a) *Given volumes of different substances and their respective specific gravities.*

Let v_1, v_2, v_3...be the volumes and ρ_1, ρ_2, ρ_3...the specific gravities respectively of the different substances in the mixture.

Mass of the first substance $= v_1\rho_1$

„ second „ $= v_2\rho_2$

.....

Mass of the mixture $= v_1\rho_1 + v_2\rho_2 + \ldots\ldots$

Let ρ be the common specific gravity.

If the total volume of the mixture is v $(= v_1 + v_2 + \ldots)$, the mass of the mixture is also equal to $v\rho$.

$$\therefore v\rho = (v_1 + v_2 + \ldots)\rho$$
$$= (v_1\rho_1 + v_2\rho_2 + \ldots)$$
$$\text{or } \rho = \frac{(v_1\rho_1 + v_2\rho_2 + \ldots)}{(v_1 + v_2 + \ldots)}.$$

(b) *Given masses and the specific gravities of different substances.*

Let m_1, m_2, m_3....be the masses and ρ_1, ρ_2, ρ_3...., the specific gravities of the different substances in the mixture.

Volume of the first substance $= \frac{m_1}{\rho_1}$

„ second „ $= \frac{m_2}{\rho_2}$

„ third „ $= \frac{m_3}{\rho_3}$

.....

$$\therefore \text{Volume of the mixture} = \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \frac{m_3}{\rho_3} + \ldots\ldots\ldots\ldots$$

If m $(= m_1 + m_2 + m_3 + \ldots)$ is the total mass of the mixture and ρ its common density,

then the volume $= \frac{m}{\rho}$.

$$\text{Hence } \frac{m}{\rho} = \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \frac{m_3}{\rho_3} \ldots.$$

$$\text{i.e., } \frac{1}{\rho} = \frac{\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \frac{m_3}{\rho_3} + \ldots.\right)}{(m_1 + m_2 + m_3 + \ldots)}$$

$$\text{or } \rho = (m_1 + m_2 + m_3 + \ldots) \Big/ \left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \frac{m_3}{\rho_3} + \ldots\right).$$

C. CENTRE OF PRESSURE

209. Thrust on a plane surface immersed in a liquid

Whenever a plane surface is immersed in a liquid, a thrust is exerted on it.

When the plane is held parallel to the surface of the liquid at a depth 'h' below its surface, the fluid pressure on the plane surface is uniform and is equal to $h \rho g$ (ρ being the density of the liquid.)

$$\therefore \text{Fluid thrust} = h \rho g a$$

(a being the area of the plane surface).

However, when the plane surface is held inclined, the pressure on the surface varies from point to point and the total thrust can be obtained as follows :

Consider a small element of area ds of the plane surface which is at a depth h below the free surface AB of the liquid.

Thrust exerted on the element $= ds. \rho. h. g$ and it is perpendicular to the area ds. i.e., to the plane surface immersed in the liquid.

Thrust on every other element of area of the surface is also similarly perpendicular to the surface and these thrusts constitute a system of parallel forces.

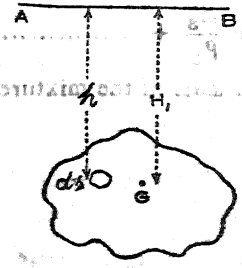


Fig. 135

Hence the thrust on the entire plane surface $= \int h. \rho. ds. g = \rho g \int h. ds.$

If the surface is of area s and if the centre of gravity of the surface, G is at a depth, H_1 below the free surface of the liquid $\int h. ds. = H_1 \cdot s.$

$\therefore$ The thrust on the plane surface $= \rho. H_1 \cdot g \cdot s.$

= Pressure at the centre of gravity $\times$ area of the surface.

Hence *the thrust executed on any plane surface immersed in a homogeneous liquid is equal to the product of the pressure at the centre of gravity of the area and the area of the surface.*

210. Centre of pressure

The total thrust on a plane surface immersed in a liquid has been shown in Art. 209 to be equal to pressure at the *c. g.* of the surface $\times$ area of the surface.

The point of application of the thrust is not in general the centre of gravity of the plane surface. The point of the surface where it acts is called the **Centre of Pressure.**

The centre of pressure of a plane surface immersed in a homogeneous liquid is defined as the point in it through which the resultant thrust of the liquid on the surface acts.

The general method for obtaining the centre of pressure of any plane surface immersed in a liquid is given below:

Let a plane surface of area s be immersed *vertically* in a liquid of density ρ.

Considering a small element of area ds in it at a depth h below the liquid surface AB, thrust exerted on $ds = h \rho g \cdot ds$.
Moment of the thrust about AB $= h. \rho. g. ds. h = h^2 \rho g ds$.

Sum of all the moments of the thrusts on such small areas

$$= \int h^2 \cdot \rho \cdot g \cdot ds \quad \ldots (1)$$

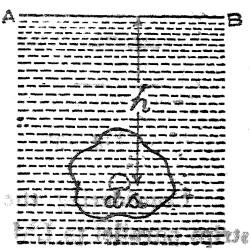


Fig. 136

This moment should also be equal to the moment of the resultant thrust ($= \int h \rho g ds$) acting on the plane surface, about AB. Let H be the depth of the centre of pressure below AB.

$\therefore$ Moment of the resultant thrust about AB

$$= H \int h \rho g \cdot ds \quad \ldots (2)$$

$\therefore H. \int h \rho g ds = \int h^2 . \rho g . ds$

$$\text{or } H = \frac{\int h^2 \rho g ds}{\int h \rho g ds} = \frac{\int h^2 . ds}{\int h . ds} = \frac{\int h^2 . ds}{H_1 . s}$$

where s is the area of the surface and H_1 is the depth of the centre of gravity of the area below the free surface.

The depth of the centre of pressure below the liquid surface can be obtained using this relation. It is seen that it does not depend upon the density of the liquid but only upon the shape of the area.

[But $\int h^2 . ds = K^2 . s$ [K being the radius of gyration of the surface of area s about the line AB where its plane cuts the surface of the liquid).

If $\int h \cdot ds = H_1 \cdot s$ where H_1 is the distance of the centre of gravity of the area from AB, then

$$H = \left[\frac{K^2 \cdot s}{H_1 \cdot s} = \frac{K^2}{H_1} \cdot \right]$$

This result is also true if the plane surface is inclined.

211. Centre of pressure of a rectangular lamina with one side in the surface

Let ABCD be a plane rectangular lamina immersed *vertically* in a liquid of density ρ with AB in the surface of the liquid.

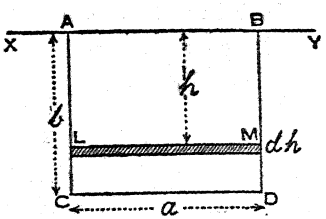


Fig. 137

Let AB = CD = a and AC = BD = b.

Consider a thin strip LM of width dh at a depth h below AB and parallel to it.

Area of the strip = $a \cdot dh$.

Pressure at any point on the strip = $h \rho g$.

∴ Thrust on the strip = $h \rho g \cdot a \cdot dh$.

Moment of the thrust about AB = $h \rho g \cdot a \cdot dh \cdot h$.

Considering the lamina to be divided into a number of thin strips parallel to LM and therefore to AB, sum of the moments of the thrusts on all the thin strips

$$= a \rho g \int_0^b h^2 \cdot dh \qquad \ldots \quad (1)$$

Resultant of the thrusts on all the strips = $\int_0^b h \rho g \cdot a \, dh$.

Let H be the depth of the centre of pressure below AB. Moment of the resultant thrust about AB

$$= H \int_0^b h \rho g \cdot a \, dh \qquad \ldots \quad (2)$$

Equating (1) and (2)

$$a \rho g \cdot H \int_0^b h \, dh = a \rho g \int_0^b h^2 \cdot dh$$

$$\text{or} \quad H = \frac{\int_0^b \cdot h^2 \, . \, dh}{\int_0^b h \, d \, h} = \frac{b^3}{3} \Big/ \frac{b^2}{2} = \frac{2}{3} b \, .$$

Owing to the symmetry of all the thin strips, the resultant thrust must act through the mid-point of all the strips, and *the centre of pressure therefore lies on the median vertical line at a depth* $\frac{2}{3}$ *b below AB.*

212. Centre of pressure in particular cases

1. Triangle with base in the liquid surface:

Centre of pressure is on the median at a depth $\frac{b}{2}$ (b being the vertical height of the median).

2. Triangle with base parallel to the surface and vertex in the surface:

Centre of pressure lies on the median at a depth $\frac{3}{4} b$ (b being the vertical height of the median).

213. Alteration in the centre of pressure with further immersion

(a) Let a plane surface ABCD be immersed vertically in a liquid with its centre of gravity G at a depth a below the surface of the liquid XY.

If s is the area of the surface, the thrust on the surface $= a \, . \, \rho \, . \, g \, . \, s$.

This thrust acts through its centre of pressure C, which is, say, at a depth b below the surface. If the lamina is now further lowered, by a depth z the pressure at every point on the plane surface is increased by $z \, \rho \, g$ and the consequent increase in thrust is $z \, \rho \, g \, s$.

The increase in thrust ($z \, \rho \, g \, s$) acts through the centre of gravity G′, since the increase is the same for every element of area of the surface.

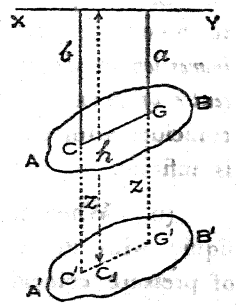


Fig. 138

$\therefore$ The total thrust on the plane is equal to $(a\,\rho\,g\,s + z\,\rho\,g\,s)$ and it acts through, say C_1, the new centre of pressure.

Now the moments of the thrust, acting at C′ and G′ (the corresponding points of C and G in the new position) respectively about XY on the surface of the liquid are together equal to the moment of the total thrust acting at C_1, about XY.

Moment about XY of the thrust acting at C′

$$= a.\,\rho.\,g.\,s.\,(b + z) \quad \ldots\ (1)$$

Moment about XY of the increased thrust acting through G′ $= z\,\rho\,g\,s\,(a + z)$ $\ldots\ (2)$

If h is the depth of the new centre of pressure C_1 below XY, the moment about XY of the resultant thrust acting through $C_1 = (a\,\rho\,g\,s + z\,\rho\,g\,s)\,h$ $\ldots\ (3)$

Equating the moments of the thrusts

$$(a\,\rho\,g\,s + z\,\rho\,g\,s)\,h = a\,\rho\,g\,s\,(b + z) + z\,\rho\,g\,s\,(a + z)$$

$$\text{or} \quad h = \frac{a\,(b + z) + z\,(a + z)}{(a + z)} = \frac{z^2 + 2az + ab}{a + z}.$$

If C′ be the position of C in the lowered position of the plane surface, depth of C_1 − depth of C′

$$= \frac{z^2 + 2\,az + ab}{(a + z)} - (b + z) = -\frac{b - a}{a + z}.z$$

which is always negative, since $b > a$.

Since C_1 is the point through which the resultant thrust passes, it should lie on G′C′ between G′ and C′. *The effect of further immersion on the position of the centre of pressure is to shift it towards the centre of gravity.* It may easily be seen that the centre of pressure coincides with the centre of gravity when the depth of immersion is infinite.

(b) When the plane surface is immersed in a homogeneous liquid with its surface parallel to that of the liquid, the centre of pressure coincides with the centre of gravity. This continues to be so even when the plane surface is lowered further without rotation.

Examples

1. (a) *Find the thrust on the rectangular end of a tank of width 100 cm. and depth 60 cm. filled with water. Also find the position where it acts.*

(b) *If one end of the tank is hinged at the bottom and held in position by bolts at each end of the top edge, find the force on each bolt.*

Thrust = Pressure at the centre of gravity × area

$$= \frac{60}{2} \times 1 \times 60 \times 100 = 18 \times 10^4 \text{ gm. wt.}$$

This is a case where the surface of the plane rectangle is in the surface.

∴ The depth of its centre of pressure below the surface

$= \frac{2}{3} \times \text{depth} = \frac{2}{3} \times 60 = 40$ cm.

∴ The thrust acts at the centre of pressure which is at a depth of 40 cm. below the surface.

The centre of pressure is 40 cm. below the surface and therefore 20 cm. above the hinge.

Taking moments about the hinge

$F \times 60 = 18 \times 10^4 \times 20$

or $F = 6 \times 10^4$ gm. wt.

∴ Force on each bolt $= 3 \times 10^4$ gm. wt.

2. *The densities of two liquids are as 3 : 5. The lighter liquid rests on the heavier to a depth of 3". A square of side 6" is immersed vertically into the liquids with one side in the free surface. Find the ratio of the thrusts on the two portions.*

Let the densities be 3ρ and 5ρ

Thrust on the upper part = pressure at the c. g. of the upper part × area

$= \frac{1}{2} \times 3 \times 3\rho \times 6 \times 3$ gm. wt.

$= 81\cdot\rho$ gm. wt. ... (1)

Thrust on the lower part = Pressure at the c. g. of the lower part × area

$= (3 \times 3\rho + \frac{3}{2} \times 5\rho) \times 6 \times 3$

$= 227\cdot\rho$ gm. wt. ... (2)

Ratio of the thrusts $= \frac{297\cdot\rho}{81\cdot\rho} = 3.67$.

D. THRUST ON A CURVED SURFACE

214. Consider a curved surface held in a fluid. The thrusts on the different elements, into which the surface may be imagined to be divided, act normal to the respective elements. Consequently the forces will be in different directions.

Each one of these thrusts (whatever may be its direction) may be resolved in the vertical and horizontal directions. The vertical components of the thrusts will all be parallel to one another and their resultant is called the **resultant vertical thrust** on the curved surface.

The horizontal components, however, will not be generally parallel to one another. They can be resolved into components along two mutually perpendicular directions, say, Ox and Oy respectively. Let X and Y represent the resultants of all the components obtained parallel to Ox and Oy respectively. The resultant of X and Y is called the **resultant horizontal thrust.**

The resultant, of the resultant vertical thrust, and the resultant horizontal thrust, gives the **total thrust** on the surface.

215. Resultant vertical thrust

Imagine a curved surface ABCD in a homogeneous liquid.

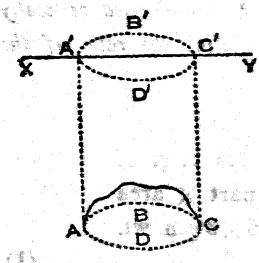


Fig. 139

Let a vertical line AA′ be considered to be taken round the surface touching it along the edge ABCD and cutting the surface of the liquid XY in A′B′C′D′.

The vertical cylinder of the liquid lying on the curved surface ABCD is called the **superincumbent liquid.**

The vertical forces acting on this superincumbent liquid are

(a) the weight of the superincumbent liquid over ABCD which acts vertically downwards, and

(b) the resultant vertical thrust on the curved surface ABCD which acts vertically upwards.

For equilibrium, the two forces must be equal, opposite and act in the same vertical line.

Hence *the resultant vertical thrust on the surface ABCD in the liquid is equal to the weight of the superincumbent liquid on it and it acts through the centre of gravity of the superincumbent liquid.*

216. Vertical thrust on the curved surfaces of a container

Let the liquid be contained in a vessel with curved surfaces as shown in Fig 140. Consider any part of the curved surface ABC. This part is subjected to a resultant upward thrust which can be obtained as follows:

Imagine the liquid surface to extend above ABC. Making the same construction as above, a cylinder ACBB′C′A′ will be obtained resting on the surface ABC, which becomes the superincumbent liquid.

The resultant vertical thrust on ABC is equal to the weight of the superincumbent liquid ACBB′C′A′.

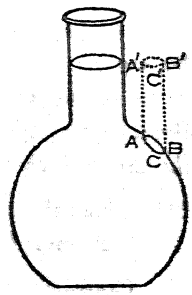


Fig. 140

Problem: Show that the resultant vertical thrust on the sides of a conical vessel filled with a liquid is equal to $\frac{2}{3}$ of the thrust on its base.

Let a liquid of density ρ be contained in a conical vessel ABC of height h and let r be the radius of the base. Consider that a cylinder BB′C′C is constructed on the base BC with the same height as that of the conical vessel.

The vertical thrust on the curved surface
= weight to the superincumbent liquid that would rest on the sides.

$= \pi a^2 h \cdot \rho g - \frac{1}{3} \pi a^2 h \rho g = \frac{2}{3} \pi a^2 h \rho g.$

But the thrust on the base $= \pi a^2 h \rho g$.

∴ The vertical thrust on the curved surface $= \frac{2}{3}$ the thrust on the base.

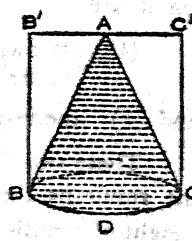


Fig. 141

217. Resultant horizontal thrust.

Let ABC be any curved surface in a homogeneous liquid.

Let LM be any vertical plane inside the liquid. Imagine a horizontal line AA′ to be taken round the surface touching it along ABC, cutting the vertical plane in A′B′C′.

The forces acting in the horizontal direction on the liquid enclosed in ABCA′B′C′ are

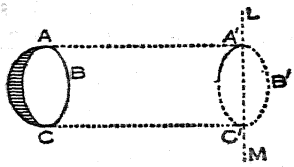


Fig. 142

(a) the resultant horizontal thrust on ABC

and (b) the thrust on the plane area A′B′C′, acting through its centre of pressure.

For equilibrium, the two forces must be equal and opposite. Hence *the resultant horizontal thrust on any given surface in contact with a fluid in any chosen direction is equal to the thrust on the projection of the surface upon a vertical plane perpendicular to the direction chosen.*

218. Resultant thrust on a body immersed in a fluid: Principle of Archimedes

Consider a body immersed in a liquid of density ρ.

Imagine the body to be removed and the cavity, produced thereby, be filled with the same liquid. This liquid is called the **displaced liquid**; it is exposed to the same thrusts due to the surrounding liquid as the solid body.

When the same liquid fills the cavity, it is the same liquid throughout and therefore every portion of it is in equilibrium. Considering the equilibrium of the displaced liquid, the forces acting on it are

(1) its own weight acting vertically downwards through its centre of gravity.

and (2) the resultant thrust due to the rest of the liquid on it.

These two forces must be therefore equal and opposite. Hence the resultant thrust acting on the displaced liquid is equal to the weight of the displaced liquid and it acts vertically upwards through the centre of gravity of the displaced liquid.

Thus the resultant thrust acting on any solid body immersed partly or wholly in any liquid is equal to the weight of the displaced liquid and it acts vertically upwards through the centre of gravity of the displaced liquid. This is the **'Principle of Archimedes'**.

Consequently, the weight of the solid body is reduced apparently whenever it is in a liquid and the apparent loss of weight is equal to the weight of the displaced liquid, i.e., equal to the weight of its own volume of the liquid.

This discussion holds good in the case of a gas also and Archimedes' principle is applied to all fluids.

219. Corrections for buoyancy

Air is a fluid. A body according to the principle of Archimedes, loses a portion of its weight when weighed in air. Consequently the weight of a body in air is less than what it would be when weighed in vacuum. The correction for the weight of a body due to the fluid thrust of air on the body is called the **buoyancy correction.**

Let ρ be the density of the body,

σ the density of air

and ϕ the density of the set of standard weights used for weighing in a balance.

Let W_1 be the true weight of the body in vacuum.

Volume of the body $= \frac{W_1}{\rho}$.

Force of buoyancy on it when weighed in air

= Weight of an equal volume of air

$= \frac{W_1}{\rho} . \sigma$.

$\therefore$ Apparent weight in air $= \left(W_1 - \frac{W_1}{\rho} . \sigma \right)$.

Let a set of standard weights whose total weight in vacuum is W_2 be found to balance the body in air.

Volume of the weights $= \frac{W_2}{\phi}$.

Force of buoyancy on the weights $= \frac{W_2}{\phi} \cdot \sigma$.

$\therefore$ Apparent weight of the set of weights in air

$$= \left(W_2 - \frac{W_2}{\phi} \cdot \sigma \right).$$

Since the body and the weights balance one another in air,

$$W_1 - \frac{W_1}{\rho} \cdot \sigma = W_2 - \frac{W_2}{\phi} \cdot \sigma$$

$$\text{or } W_1 \left(1 - \frac{\sigma}{\rho} \right) = W_2 \left(1 - \frac{\sigma}{\phi} \right)$$

$$\text{or } W_1 = \frac{W_2 \left(1 - \frac{\sigma}{\phi} \right)}{\left(1 - \frac{\sigma}{\rho} \right)} = W_2 \left(1 - \frac{\sigma}{\phi} + \frac{\sigma}{\rho} \right) \text{ approximately.}$$

E. LAW OF TRANSMISSIBILITY OF FLUID PRESSURE

When a change of pressure is produced at any point in a fluid at rest it is transmitted equally to all parts of the fluid, and the same change of pressure is brought about at every other point of the fluid.

This law can be well demonstrated by taking a rubber ball having a number of holes. If the ball is filled with water and squeezed, water is seen to come out of the holes with equal force illustrating the principle.

Proof: Let M and N be any two points in a homogeneous liquid at rest.

Consider a thin cylinder of liquid constructed around the line M and N as axis having an area of cross-section a.

Let P_1 and P_2 be the pressure at M and N respectively. The various forces acting on the cylinder are

(1) $P_1 \cdot a$, acting at M

(2) $P_2 \cdot a$, acting at N

and (3) W, the weight of the liquid in the cylinder which acts vertically downwards from its centre of gravity.

For equilibrium, the algebraic sum of the forces resolved in the direction of MN should be zero.

$\therefore P_1 a - P_2 a - W \cos \alpha = 0 \ldots (1)$

(α being the angle which the axis MN makes with the vertical).

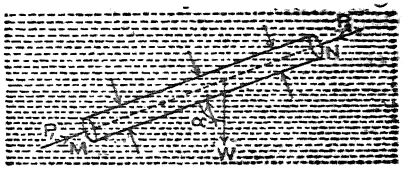


Fig. 143

Let an extra pressure p_1 be applied at M and let p_2 be the consequent increase of pressure at N.

Again for equilibrium in the direction of MN

$(P_1 + p_1)\, a - (P_2 + p_2)\, a - W \cos \alpha = 0 \quad \ldots (2)$

From (1) and (2) $p_1 a - p_2 a = 0$ or $p_1 = p_2$.

Hence *an increase of pressure at any point M is equally communicated to any other point N in the liquid.* This is **Pascal's law.** The same law holds good in the case of gases also.

220. Hydraulics and hydraulic machines

Hydraulics is the engineering science in which the principles of Hydrostatics and Hydrodynamics (especially those connected with water running through pipes and channels) are applied. Machines which work on these principles are known as **hydraulic machines.** The Hydraulic Press (or Bramah Press), the hydraulic lift and the hydraulic pump and several other hydraulic machines are widely used at present. *They offer several advantages like flexibility of speed and feed control.*

221. Hydraulic Press (or Bramah Press)

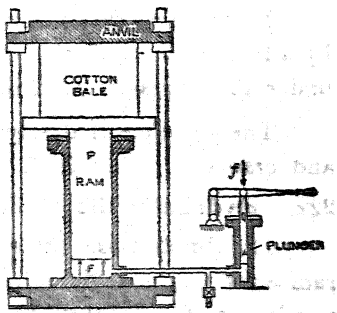


Fig 144

The hydraulic press consists of two cylinders, one of them having a much larger diameter (2A) than the other one (having a diameter $2a$) They are connected at their lower ends by means of a tube. The cylinders are filled with water and fitted with water-tight pistons P and Q respectively. The piston P fitted in the larger cylinder is called the **ram** while Q

fitted in the smaller cylinder is called the **plunger.** To use the machine in compressing materials such as cotton bales, an anvil (or a cross-bar) is fixed horizontally above the ram leaving some space between the two for keeping the material to be compressed.

Action of the Press

Let the plunger Q be lowered with a certain force f.

The increase of pressure just below the plunger $= \frac{f}{a}$.

The increase of pressure is, in accordance with Pascal's law, communicated to every point of the liquid in the larger cylinder by the flow of water under high pressure from the smaller cylinder to the larger.

$\therefore$ Increase at a point just below the ram

$$= \frac{f}{a} \text{ and it acts upwards.}$$

Hence the force upwards acting on P $= \frac{f}{a} \times$ A. Thus the ram P is forced upwards with a force equal to $\frac{f}{a} \times$ A. The cotton bale or any other material introduced between the ram and the anvil, is pressed between the two with this force.

It is obvious that by reducing a and increasing A, larger forces of compression can be obtained.

In practice, the smaller cylinder and the plunger Q are replaced by a feed pump operated by a steam engine. The pump sends water under high pressure into the larger cylinder containing the ram P.

The principle of the hydraulic press is applied in hydraulic lifts and cranes.

222. Hydraulic lift

The simplest and the easiest form of the lift consists of a sliding ram working in a cylinder. The lift or cage is supported on the ram and is therefore pushed up or lowered down along with the ram, when the pressure is increased or decreased. This type of the lift is called the **direct-acting lift.** The weight of the ram together with the weight of the lift and the load in it is equal to the total force of compression on the ram.

In the modern types, the lift is suspended from a wire rope which passing over a pulley, is connected to the hydraulic ram. The lift ascends when the ram is lowered. This arrangement has the advantage of keeping the ram and the lift move without any slip.

In practice, the wire ropes are not directly connected to the ram. They are passed over a multi-sheaved system of pulleys to increase the velocity ratio of the lift. The lift is arranged to run between parallel guides of hard wood or round steel and sliding balance weights are provided to nearly balance the weight of the lift so that the load carried by the lift can be the maximum possible.

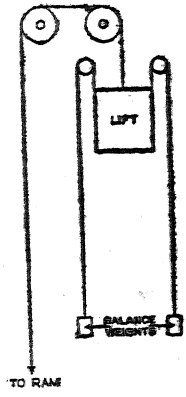


Fig. 145

223. Hydraulic pump

The hydraulic pump consists of a cylinder A fitted with a piston B which moves to and fro inside it by means of a sliding rod C operated by a steam engine. The cylinder is provided with two valves E and F which rise and fall vertically. When they are in raised positions, passages K and L are opened for water to run through.

When the piston moves towards right as in Fig. 146, water enters A through the opening M, past the valve K. At the same time water on the left hand side of the piston is being expelled under high pressure through a passage G and then into the delivery pipe H.

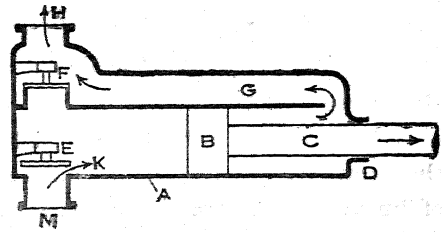


Fig. 146

When the piston moves towards the left, the valve, E closes and the valve F rises providing a passage at L. The water is forced through the opening at F into the delivery pipe H. Part of the water forced into the space above F now enters also to the right hand side of the cylinder which will be forced through H when the piston moves towards the right.

Thus water is forced into the delivery pipe when the piston moves towards the right or the left.

224. Hydraulic Jack

The ram itself may also be used as a jack to lift heavy weights. The load to be lifted is placed on the ram. When the water at high pressure is forced into the cylinder, the ram is raised and load lifted. By suitable arrangements the ram can be fixed at any desired height.

When the outlet connected to the cylinder is opened, water from the cylinder is withdrawn, the ram slides down and the load on it descends.

F. FLOATING BODIES

225. We learn from Archimedes' principle that when a solid body is lowered into a liquid, its weight W is opposed by the resultant upward thrust of the liquid (or the force of buoyancy) acting on the body, which is equal to the weight of the displaced liquid. Left to itself, the body may sink in the liquid or remain in equilibrium in the liquid or rise to the surface and float on the liquid according as the density of the body is greater than, equal to or less than the density of the liquid.

Thus when the density of the liquid is more than that of the body, the body floats on the surface of the liquid. The forces that act on the floating body in the vertical direction are :

(1) its own weight W which acts vertically downwards through its centre of gravity, and

(2) the force of buoyancy which acts vertically upwards through the centre of gravity of the displaced liquid, i.e., through the centre of buoyancy. Hence the **conditions of equilibrium** for a floating body may be stated thus :

(1) the weight of the body and the weight of the displaced liquid should be equal ; and

(2) the centre of gravity of the body and the centre of buoyancy of the displaced liquid should lie in the same vertical line.

226. Hydrometers

A hydrometer is an instrument that measures the specific gravity of liquids, when it is floated on the latter. There are two types of hydrometers which are commonly used :

(1) the variable immersion hydrometer called the *common hydrometer*, and

(2) the constant immersion hydrometer, called *Nicholson's hydrometer*.

227. The common hydrometer

It consists of a long uniform glass tube AB provided with a sufficiently large bulb C at its lower end B. The bulb is loaded with a sufficient amount of lead shot or mercury *to make it float vertically in different liquids.* The long stem AB is graduated so that the specific gravity of a liquid on which it is floated, can be read off directly. Hydrometers having different ranges are provided for use with different liquids.

The weight of the hydrometer is so adjusted that it floats vertically in all the liquids, the specific gravities of which fall within its range. The weight of the hydrometer remains constant and consequently it displaces equal weights of different liquids when floated on them.

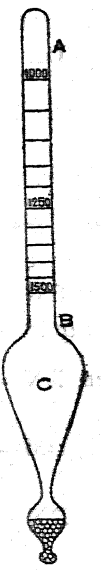


Fig. 147

Let V be the total volume of the hydrometer.

a the area of cross-section of the stem AB,

l_1 the length of the stem exposed when floated on water,

l_2 the length of the stem exposed when floated on a liquid of density ρ_1,

l_3 the length of the stem exposed when floated on a liquid of density ρ_2

and so on.

Then the volumes of different liquids displaced are

$(V - l_1 a)$; $(V - l_2 a)$; $(V - l_3 a)$; ...

$\therefore$ The weights of different liquids displaced are

$(V - l_1 a) \times 1$; $(V - l_2 a)\,\rho_1$; $(V - l_3 a)\,\rho_2$; ...

and they are equal to one another.

Hence $(V - l_1 a) = (V - l_2 a)\,\rho_1 = (V - l_3 a)\,\rho_2$.

Fig. 148

It follows that $V = \dfrac{(\rho_1 l_2 - l_1)a}{\rho_1 - 1}$,

$$\rho_1 = \frac{V - l_1 a}{V - l_2 a}$$

and $$\rho_2 = \frac{V - l_1 a}{V - l_3 a}.$$

Thus knowing, V, l_1 and l_2 and the densities of the corresponding liquids, the stem may be graduated.

The graduations on the stem get closer when the densities of the liquids increase uniformly as can be seen below.

Let $V = la$. The hydrometer is then equivalent to a long cylinder of length l and area of cross-section a. Now let the stem AB be prolonged down to D (replacing the bulb) so that AD = l Fig. 148).

If L and M are the marks on the hydrometer up to which it floats in water and in a liquid of density ρ, then

$$DL \times 1 = DM \times \rho$$

$$\text{or} \quad \rho = \frac{DL}{DM}$$

$$\therefore \quad DM = \frac{DL}{\rho}.$$

Hence the distance of M from D is inversely proportional to the density of the liquid.

If it floats to marks N, O, P, ...

in liquids of densities ρ_1, ρ_2, ρ_3, ...

then $DN = \dfrac{DL}{\rho_1}$; $DO = \dfrac{DL}{\rho_2}$; $DP = \dfrac{DL}{\rho_3}$; ....

and $$DM - DN = DL\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right),$$

$$DN - DO = DL\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right),$$

$$DO - DP = DL\left(\frac{1}{\rho_2} - \frac{1}{\rho_3}\right),$$

.....

If $(DM - DN) = (DN - DO) = (DO - DP) = \ldots\ldots$, then

$$\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \left(\frac{1}{\rho_2} - \frac{1}{\rho_3}\right) = \ldots$$

Thus if the points M, N, O, P, ... on the stem are marked so that their distances from D are in arithmetic progression, the densities ρ, ρ_1 ρ_2, ρ_3, ... of the corresponding liquids are in harmonic progression.

228. Nicholson's hydrometer

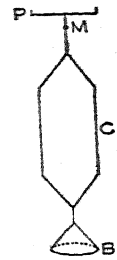


Fig. 149

It consists of a hollow cylindrical body which is provided with tapering ends. It is connected to a scale-pan P at the top by means of a long narrow stem and to a loaded bucket B at the lower end. A mark M is made on the stem to indicate the constant level to which it will have to be sunk in different liquids. The hydrometer is sunk in any liquid to the mark M by adding sufficient weights on the pan P until it sinks to the mark.

Let W be the weight of the hydrometer,

W_1 the additional weight placed on the pan to lower it to the mark M in water, and

W_2 the additional weight placed on the pan to sink it to the same mark M when floated on liquid of density ρ.

Since it floats to the same mark M on both water and liquid, the volumes of water and liquid displaced are equal. If V is the volume of water or liquid displaced, then

$$(W + W_1) = V \times 1,$$
$$(W + W_2) = V \times \rho.$$
$$\therefore \frac{\rho}{1} = \rho = \frac{W + W_2}{W + W_1}.$$

Determining W, W_1 and W_2 the specific gravity of any liquid can be determined.

229. Density of a solid with Nicholson's hydrometer

The hydrometer requires a definite weight to sink to the mark M in water. This weight includes the weight of the hydrometer and the additional weights placed on the pan. Since the weight of the hydrometer is a constant, the additional weight required to be placed on the pan is also a constant.

This condition enables us to use Nicholson's hydrometer as a *rough balance* to determine the weights of small bodies as follows :

Suppose that W_1 gm. are required to be placed on the pan to sink it to the mark in water. If on placing the given body on the pan, W_2 gm. are required to be put on the pan to lower it again to the mark M, then

$$W_1 = W_2 + \text{the weight of the body.}$$

$$\therefore \text{The weight of the body} = W_1 - W_2.$$

If now the given body is transferred to the bucket and the additional weight W_3 is similarly obtained, the weight of the body in water $= (W_1 - W_3)$.

$\therefore$ Loss of weight in water

$$= \text{Wt. of an equal volume of water}$$

$$= (W_1 - W_2) - (W_1 - W_3) = W_3 - W_2.$$

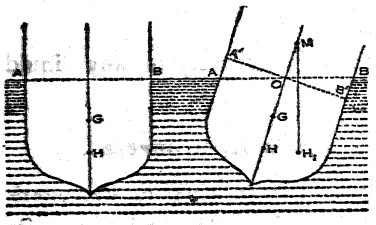
Hence the specific gravity of the body $= \dfrac{(W_1 - W_2)}{(W_3 - W_2)}$

Determining W_1, W_2 and W_3 the specific gravity of the body can be determined.

230. Metacentre of a floating body

When a body floats on the surface of a liquid, it has been shown that the centre of gravity of the body and the centre of buoyancy of the displaced liquid lie in the same vertical line.

The section in which the plane of the surface of the liquid intersects the floating body is called the **plane of floatation.**



(a) Fig. 150 (b)

When the floating body moves on the surface of the liquid, the centre of buoyancy also gets displaced. The surface, traced out by the centre of buoyancy as the body moves, displacing equal volumes of the liquid, is called the **surface of buoyancy.**

If a floating body be displaced slightly so that the volume of the liquid that it displaces remains the same, the point at which the

vertical through the centre of buoyancy in the displaced position meets the median line, (i.e., the line joining the centre of gravity of the body and the original centre of buoyancy) is called the **Metacentre.**

Let AB be a body floating on the surface of the liquid.

Let G be the centre of gravity of the body,

H, the centre of buoyancy of the displaced liquid and H_1 the centre of buoyancy of the displaced liquid in the new position in which the body displaces an equal amount of the liquid.

If now the vertical line through H_1 meets the line passing through G and H at M, M is the metacentre. The distance GM is called the **metacentric height.**

231. Stability of a floating body

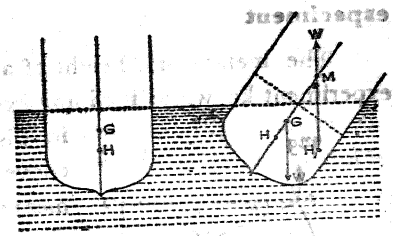
It is possible that the line through H_1 may meet the line GH above G or below G.

(a) Consider that the metacentre M lies above G as in Fig. 151-b.

The floating body is now subject to two forces :

(1) its own weight acting vertically downwards through G and

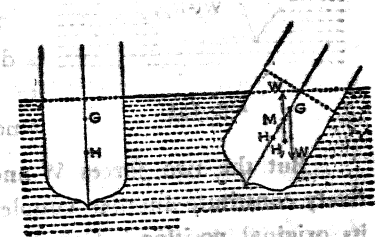
(2) the force of buoyancy (also equal to W acting vertically upwards through H_1.



(a) Fig. 151 (b)

These two equal and opposite forces constitute a couple, *the moment of which tends to restore the body back to its undisturbed position.* The equilibrium of the floating body is then **stable.**

(b) Next, consider that the metacentre lies below G as in Fig. 152-b. The weight of the body W acting vertically down through G and the force of buoyancy W acting vertically up through H_1 constitute a couple, *the moment of which tends*



(a) Fig. 152 (b)

to displace the body further away from the undisturbed position. The equilibrium of the body is therefore **unstable.**

Thus, *a floating body is in stable or in unstable equilibrium according as the metacentre M lies above or below G.*

232. Distance between the centre of buoyancy and metacentre

Let AB and A'B' be the planes of floatation of a body in the orignal and in the (slightly) displaced positions. (Fig. 150–b).

The two planes of floatation meet along a line through O perpendicular to the plane of the paper. This line perpendicular through O is called the **axis of rotation.**

Let AK^2 be the moment of inertia of the plane of floatation about the axis of rotation and V the volume of the liquid displaced.

It can be shown mathematically that $HM = \frac{AK^2}{V}$.

233. Determination of the matacentric height: Inclining experiment

The metacentric height of a ship can be determined by a simple experiment known as inclining experiment. The experiment consists in displacing a known weight through a convenient distance across the deck and measuring the angle through which the plane of floatation of the ship is caused to deflect.

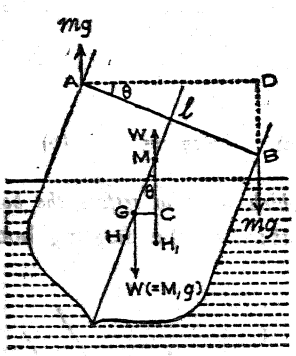


Fig. 153

Let a heavy mass *m* be moved from A to B through a distance *l* across the deck of the ship of mass M_1.

The shifting of the weight (*mg*) through a distance *l* across the deck is equivalent to applying a deflecting couple which tends to turn over the ship.

But the two forces W and W acting through G and H_1 respectively constitute another couple which tends to restore the ship to its original position. Consequently, the ship attains equilibrium in a position wherein the two couples balance each other.

Let the plane of floatation be turned through an angle θ.

Deflecting couple $= mg.\ l \cos\theta$

Restoring couple $= M_1 g.\ GM \sin\theta$

$\therefore\ mgl \cos\theta = M_1 g.\ GM \sin\theta$

or $GM = \dfrac{mgl \cos\theta}{M_1 g \sin\theta} = \dfrac{ml}{M_1\theta}$ (when θ is small).

Thus knowing M and m and measuring θ using a plumb line the metacentric height GM can be determined.

It may be noted that *the metacentric height changes with a change in the load contained in the ship. The experiment therefore gives the height GM for the ship with that particular load in it at the time of the experiment.*

234. Metacentre of a floating body with a spherical bottom

When a body floats on the surface of a liquid, the thrusts acting on the small elements (into which the surface of the body under the liquid may be supposed to be divided) are normal to the surface at every point. The lines of action of all the thrust therefore pass through C, the centre of curvature of the spherical surface. The resultant of all these thrusts, which acts vertically upwards through the centre of buoyancy therefore passes through C, in the disturbed or undisturbed positions of the float.

But in the equilibrium position, before being disturbed, the centre of gravity of the body G and the centre of buoyancy of the displaced liquid H lie in the same vertical line. The line GH therefore passes through C. If the body is now slightly displaced, the resultant thrust acts vertically upwards through the new centre of buoyancy H_1 and a vertical line through H_1 passes through C. Thus the line through GH and the vertical line through H_1 meet at C. The centre of curvature C is therefore the metacentre of the body with a spherical bottom.

G. ATMOSPHERIC PRESSURE AND BAROMETERS

235. The envelope of air that surrounds the earth is called the *atmosphere* and the *pressure* exerted by it at any point on the surface of the earth, the *atmospheric pressure.*

Barometer: A barometer is an instrument that measures the atmospheric pressure. The first barometer was constructed by an Italian scientist, Torricelli in 1643. A simple barometer may be set up in the laboratory as follows:

A glass tube, about 1 cm. in diameter and 90 cm. in length, having stout walls, is closed at one end. It is filled completely with clean dry mercury, *care being taken to see that no air bubbles are left inside.* The tube is then closed with the thumb and inverted into a trough of mercury. On removing the thumb, the mercury in the tube is seen to fall and stand at a certain height. The vertical height of the mercury in the tube is seen to remain unaltered when the tube is raised or lowered or inclined to the vertical (Fig. 154). The space above the mercury column in the tube is vacuum and it is known as the *Torricellian vacuum.*

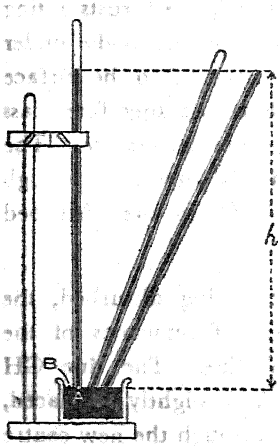


Fig. 154

The height of the mercury column above the surface of mercury in the trough gives the atmospheric pressure in cm. of mercury as can be seen below.

The pressure at a point B on the surface of mercury outside the tube = the pressure at a point A inside the tube and in the same horizontal plane as B. But the pressure at B = the atmospheric pressure and the pressure at A = the pressure due to the mercury column of height, say, h cm. in the tube.

∴ Atmospheric pressure = the pressure due to the vertical column of mercury in the barometer tube.

Atmospheric pressure is generally measured by the height of the mercury barometer in cm. instead of in absolute units.

236. Fortin's barometer

Fortin's barometer is a standard barometer used for measuring the atmospheric pressure. It consists of a cistern of mercury the lower part of which is made of leather. A screw S_1 is provided below the bottom of the cistern by working which the level of

mercury in it can be raised or lowered. There is an ivory index P, fixed to the top of the cistern, the pointed end of which corresponds to the zero of the barometer scale and the surface of mercury in the cistern is to be made to coincide with the pointed end before readings are taken. The glass tube containing mercury column is surrounded by a brass tube having a vertical slot at the top, through which the column of mercury is visible. Scales one in cm. and the other in inches (the zeros of which coincide with the pointed end P) are engraved on the two sides of the opening. A vernier bit V moves along the scales and it can be worked by means of another screw S_2. A similar brass bit at the back with its lower edge at the same level as that of the vernier, moves with the vernier in a similar slot behind the tube. This second bit at the back helps in taking readings without the error of parallax. Milk-white glass plates are provided behind the cistern and behind the openings at the top to ensure better visibility during adjustments.

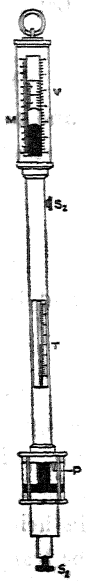


Fig. 155

To obtain the atmospheric pressure, the following adjustments are made in order:

(1) The mercury level in the cistern is raised or lowered by working the screw S_1 so as to make it just touch the pointed end of the pin P. This will be the case when the pointed end of P just touches its image in the mercury surface *without producing a depression* at the point of contact.

(2) The vernier is raised sufficiently by working the screw S_2 so that the meniscus of the mercury is seen clearly. Keeping the eye in the line of the lower edges of the vernier and the brass bit behind, the lower edge of the vernier is made to coincide with the meniscus of mercury without any parallax.

(3) The readings on the scales are noted. They give the atmospheric pressure at the time of the experiment in cm. or inches of mercury. The readings obtained from the barometer will have to be corrected for the expansion of mercury and also for the expansion of the brass tube on which the scales are etched.

Let H be the observed height of mercury at the room temp. $t°$ C and H_o the corrected height.

If a and c are the coefficients of the linear expansion of brass and the volume expansion of mercury respectively and the scale be correct at 0°C., it can be shown that

$$H_o = H \left\{ 1 - (c - a)t \right\}.$$

The atmospheric pressure at a place varies from time to time and it is not a constant.

237. Variation of atmospheric pressure with height

Since the pressure of the atmosphere at a point is equal to the weight of a vertical column of air of unit cross-section standing above the point and since the height of this column decreases as the point is taken at a higher level, the pressure at a place decreases with its height above the earth. Thus the pressure of the atmosphere will be less on the top of a mountain and more at the bottom of a mine than at sea level.

The **standard or normal atmospheric pressure** is the pressure of the atmosphere when the height of the mercury barometer is 76 cm. at 0°C at sea level in latitude 45° (where g = 980·6 cm./sec.^{2}).

The difference in pressure at two points can be obtained in terms of their height above the earth as follows:

Let P and Q be two points in the same vertical line at heights h and $(h + dh)$ above the surface of the earth and let p and $(p + dp)$ be the respective pressures at the two points.

Join PQ and imagine a thin cylinder constructed around the line with PQ as axis.

If ρ is the density of the air in the cylinder, weight of the air in it $= a \cdot dh \cdot \rho \cdot g$ (a being the area of cross-section of the cylinder).

The column of air in the cylinder is subject to the following forces:

(1) downward thrust $[(p + dp)\, a]$ on the upper surface of the cylinder,

(2) upward thrust, $(p.\ a.)$ on the lower surface of the cylinder and

(3) the weight of the air in the cylinder, $(a \cdot dh \cdot \rho g)$ which acts vertically downwards.

For equilibrium:

$$p \cdot a = (p + dp)\, a + a \cdot dh \cdot \rho \cdot g$$

or $dp = -\rho \cdot g \cdot dh$

(the $-ve$ sign indicates that the pressure decreases as the height increases).

But according to Boyle's law, the pressure varies as the density of the gas, when the temperature is the same

Let $\rho = k \cdot p$ (where k is a constant)

$$\therefore dp = -k \cdot p \cdot g \cdot dh$$

or $$\frac{dp}{p} = -k \cdot g \cdot dh \qquad \ldots\ (1)$$

Integrating (1), $\log_e p = -k \cdot g \cdot h + C$.

Let the pressure on the surface of the earth be p_o so that when $h = 0$, $p = p_o$.

$$\therefore \log_e p_o = 0 + C$$

or $C = \log_e \cdot p_o$.

Hence $\log_e \cdot p = -k \cdot g \cdot h + \log_e p_o$

$$\text{or } \log_e \frac{p}{p_o} = -k \cdot g \cdot h.$$

$$\therefore \frac{p}{p_o} = e^{-kgh} \quad \text{or} \quad p = p_o \cdot e^{-kgh} \qquad \ldots\ (2)$$

(when the temperature is the same throughout the atmosphere, i.e., when the atmosphere is in isothermal equilibrium).

Since $p \propto \rho$, the density of air ρ_o on the surface of the earth is related to the density of air at any height h, by the relation

$$\rho = \rho_o \cdot e^{-kgh}.$$

(a) Let p_1, p_2, p_3.....be the pressures of the air at heights h_1, h_2, h_3.....above the surface of the earth.

Then $\log_e \frac{p_1}{p_o} = -k g h_1$

$$\log_e \frac{p_2}{p_0} = -kgh_2$$

$$\therefore \log_e \frac{p_1}{p_2} = -kg.(h_1 - h_1) \quad \text{.... (3)}$$

$$\text{Similarly, } \log_e \frac{p_2}{p_2} = -kg\,(h_2 - h_3)$$

.....

It follows therefore that when $(h_1 - h_2) = (h_2 - h_3) = \ldots$

$$\frac{p_1}{p_2} = \frac{p_2}{p_3} \quad \ldots \quad \ldots$$

Thus *as the altitude increases in arithmetic progression, the corresponding pressure at these altitudes varies in geometrical progression.*

(b) For determining the difference in height of two places, relation (3) above can be used;

$$\text{i.e., } \log_e \frac{p_1}{p_2} = -kg\,(h_1 - h_2)$$

But for small difference of height only, relation (3) can be employed with sufficient accuracy. It should be noted however that this relation has been obtained on the supposition that the temperature throughout is constant. It is not the actual case however.

Substituting for $k = \frac{\rho}{p}$ (p and ρ being the pressure and the corresponding density of air at a given temperature),

We have $\frac{p}{\rho} = \frac{0{\cdot}001293}{76 \times 13{\cdot}6 \times 980}$ and

$$\log_e \frac{p_1}{p_2} = -\frac{\rho}{p}\, g\,(h_1 - h_2).$$

238. Aneroid Barometer

Fortin's barometer is neither portable nor convenient for taking rapid readings. When high accuracy is not required, an aneroid

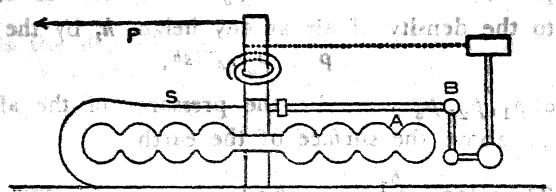


Fig. 156

barometer (i.e., a barometer without any liquid in it) is used to measure the atmospheric pressure.

It consists of a drum A, made of flexible metal and is almost exhausted of air. Its walls are corrugated so as to increase its flexibility and it is kept in position (without collapsing on account of the atmospheric pressure) by a strong spring S. Any change in the atmospheric pressure causes a slight movement of the upper face of the drum. This small movement of the face is magnified by a set of levers B and then communicated to a pointer P which moves over a scale. The aneroid barometer is therefore a direct reading instrument and it will have to be checked and calibrated from time to time comparing its observations with the readings of a standard barometer.

G. WATER PUMPS

239. Common pump

The common pump has been in use from ancient time to lift water from wells.

It consists of a cylindrical tube AB, known as the barrel. The barrel is provided with an outlet, E, at the top and is connected at its bottom to a long vertical tube, CD, leading to the water in a well. An air-tight piston P works inside the barrel. Two valves V_1 and V_2 are provided at the bottom of the barrel and in the piston respectively. Both the valves open upwards and they do not allow air or water to flow downwards.

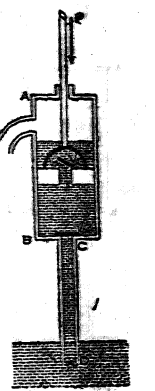


Fig. 157

When the piston is given an upward stroke, a partial vacuum is created in the barrel below the piston. The air above the piston cannot enter this space as the valve V_2 will be closed by the excess of pressure above it. The valve V_1 is forced open by the air in the tube CD and the air in CD expands into the barrel. It follows from Boyle's law that the pressure of air in CD decreases and consequently water rises into CD.

If the piston is now given a downward stroke, the air in the barrel is compressed; the valve V_1 closes and the value V_2 opens. The air in the barrel escapes out through V_2. A few strokes of the piston up and down, bring the water up to C in the leading tube CD.

An upward stroke of the piston now brings water into the barrel. A downward stroke forces the same water to go above the piston. The next upward stroke of the piston draws more water into the barrel and causes the water already above the piston to issue out of the nozzle.

It may be noted that the atmospheric pressure, acting on the water in the well, is forcing up the water through the leading tube CD into the barrel, as and when a low pressure is created in it. Hence it is essential that the *leading tube CD should not be longer than the height of the water barometer.*

240. Force pump

A force pump differs in construction from a common pump:

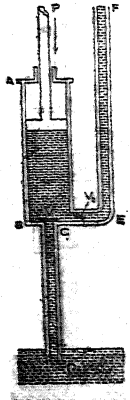


Fig. 158

(i) the outlet tube EF is provided at the foot of the barrel; (ii) there is no valve in the piston. Instead, the valve V_2 is provided at the point where the outlet tube joins the barrel and it opens into the outlet.

After the first few strokes of the piston, water rises into the tube CD up to the foot of the barrel as in a common pump. An upward stroke, thereafter, draws water into the barrel while a downward stroke forces the water up the outlet tube EF, opening the valve V_2. A force pump lifts water to a height much greater than a lift pump.

In this case also *the length of the tube CD should not be greater than the height of the water barometer,* i.e., about 34 ft.

The force pump and also the common pump cannot deliver water continuously. The force pump delivers water only during the downward stroke and the common pump delivers it only during the upward stroke. A fire-engine, which is a combination

of two force pumps, is used when a continuous stream of water is required.

241. The Fire-engine

The fire-engine consists of a double force pump which works in such a way that, while the piston of one force pump ascends, the piston of the other one descends. Further, there is an air chamber A into which the water enters from the two pumps. The air in the chamber is compressed by the water and consequently serves to force the water out through the outlet continuously.

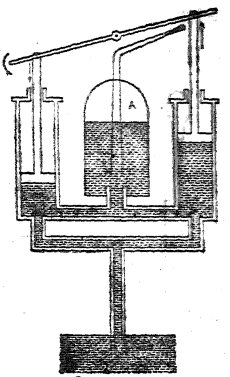


Fig. 159

242. Centrifugal pump

The centrifugal pump consists of three parts : (a) the casing (b) the impeller and (c) the spindle.

The casing is provided with an inlet I near its axis and an outlet O at its periphery. The impeller is a hollow revolving wheel provided with vanes and mounted on the spindle, the axis of which passes through the axis of the casing. The spindle conveys the necessary energy from an electric motor to run the impeller.

The pump has no suction effect and it therefore begins to function only when the casing is filled with water. When once the casing is full and the impeller begins to rotate, the pressure near the axis of rotation falls below that outside and the outside air forces water into the casing through the inlet provided near the axis. A non-return valve is sometimes provided in the inlet so that water which once enters the casing is stored there itself. Or, the entire pump is drowned under water so that water will be present near the inlet itself.

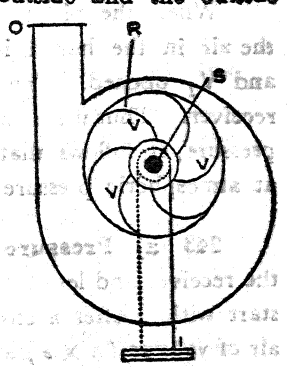


Fig. 160

This pump works on the principle of centrifugal force. When the impeller is rotated, the water between the vanes

is also rotated rapidly. Consequently, due to the centrifugal force developed inside, water is flung on the sides of the casing with a large amount of kinetic energy. When it is thus flung, a large pressure is produced in the water which enables the water to be raised up the outlet tube. No valves are used in the construction of the pump and it can therefore be even when the water is not perfectly clean.

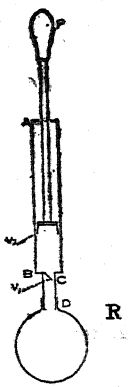


Fig. 161

243. Compression pump or Inflator

A compression pump is used to increase the pressure of air in any container. A cycle pump and a football inflator are common examples of the compression pump.

It consists of an air-tight piston P working inside a barrel, AB. A small tube CD connects the pump to the receiver R, i.e., to the vessel in which the pressure of the air is to be increased.

Two valves V_1 and V_2 are provided, one in the tube CD and the other in the piston; they open downwards. When the piston is drawn out fully, i.e., given the upward stroke, the pressure of the air in the barrel is reduced; the valve V_1 is closed and the valve V_2 opened; air from outside enters the barrel until its pressure becomes equal to the atmospheric pressure.

When the piston is lowered, i.e., given the downward stroke, the air in the barrel is compressed; the valve V_2 will be closed and V_1 opened. The air from the barrel forces its way into the receiver. Thus if v is the volume of the barrel and P the atmospheric pressure, it follows that for every downward stroke, air of volume v at atmospheric pressure P is pumped into the receiver.

243 (a). Pressure after n strokes: Let V be the volume of the receiver and let it contain air at the atmospheric pressure P to start with. After n strokes the receiver would have received into it, air of volume $(n \times v)$ at atmospheric pressure P. It would contain a certain quantity of air which would occupy a volume $(nv + V)$

at atmospheric pressure P. If P_n is the pressure of air in the receiver after n strokes, it follows from Boyle's law that

$$P_n \times V = P\,(V + nv)$$
$$\text{or } P_n = P\,(V + nv)\,/\,V.$$

244. Exhaust pump or the Common Air pump

Exhaust pump or air pump is an instrument used for removing air from a vessel called **receiver.**

A simple type of exhaust pump is shown in Fig. 162. It is exactly similar to a compression pump in its construction but the valves V_1 and V_2, unlike the valves in a compression pump, open outwards. P is an air-tight piston. The barrel, B, is connected to the receiver, R, by means of a leading tube; V_1 and V_2 are the valves opening outwards.

During the upward stroke of the piston, the pressure of the air in the barrel becomes less, the valve V_1 is closed on account of the excess of pressure above it and the valve V_2 opens. Air from the receiver enters the barrel.

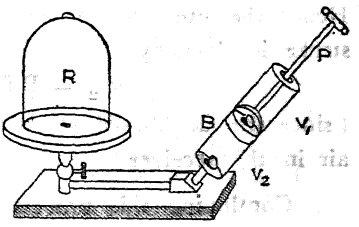


Fig. 162

During the downward stroke of the piston, the air in the barrel is compressed; the valve V_1 opens and the valve V_2 closes. The air in the barrel escapes out into the atmosphere through V_1.

Thus the net result of working the piston through a complete stroke (up and down) is to remove a certain amount of air (equal in volume to the volume of the barrel) from the receiver and thereby cause a lowering of pressure in it.

244 (a). Pressure after n strokes: Let V be the volume of the receiver and v the volume of the barrel. Let P be the pressure of air in the receiver initially.

At the beginning of the first upward stroke the volume of air in the receiver is V and its pressure, P. At the end of the first upward stroke the volume of the air becomes $(V + v)$. Consequently, if

P_1 is is the pressure of the air in the receiver now, it follows from Boyle's law that $P_1 (V + v) = PV$ or $P_1 = P \frac{V}{V + v}$.

The downward stroke removes the air from the barrel but does not alter the pressure of the air in the receiver.

Hence the pressure of the air in the receiver after the first stroke is given by $P_1 = P \frac{V}{V+v}$.

This is the pressure in the receiver at the beginning of the second stroke.

If the pressure be P_2 at the end of the second upward stroke it follows similarly, that

$$P_2 (V + v) = P_1 V$$

$$\text{or} \quad P_2 = P_1 [V/(V + v)] = P [V/(V + v)]^2$$

Hence the pressure of the air in the receiver after the second stroke is given by

$$P_2 = P [V/(V + v)]^2$$

(since the downward stroke does not affect the pressure of the air in the receiver).

Continuing this process it follows from Boyle's law that if P_n be the pressure of air after n strokes,

$$P_n = P [V/(V + v)]^n.$$

Though theoretically the pressure can be reduced to any low value by increasing the number of strokes, it is not possible actually to reduce the pressure beyond a certain stage. When the pressure is low, the valves fail to work and there is also leakage in the valves. There is usually some clearance space between the piston and the valve-end of the barrel which also reduces the efficiency of the pump.

245. Filter pump: The filter pump consists of a glass tube AB ending in a nozzle at B. The tube AB opens into another tube CD which is provided with a wide opening at C. AB and CD are fused into a larger bulb tube E. The side tube T connects the bulb tube to the receiver which is to be exhausted of air.

When water from a water tap is made to flow down the tube AB, its velocity gradually increases and becomes maximum when the water emeges out at the nozzle B. Consequently, the pressure of the air at B falls and the air from the receiver is drawn into the space near the nozzle. The water, as it flows down the tube CD, carries the air with it.

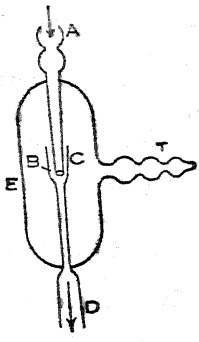


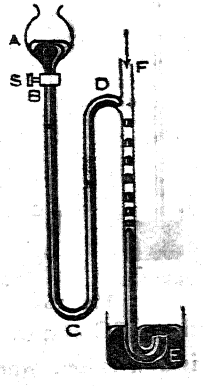
Fig. 163

The pump when worked continuously can proudce pressures of the order of 15 mm. of mercury.

246. Sprengel's mercury pump

Sprengel devised a pump using mercury by means of which low pressures of the order of 0·001 mm. of mercury were obtained. It may be interesting to note that it was only after this pump had been invented that progress in Modern Physics could take place.

The pump consists of a glass tube BCDE bent as shown in Fig. 164. The tube is connected at B to a reservoir of mercury A by means of a stop-cock, S, and the end E is dipped into a cistern of mercury. A side tube F is also provided near D for connection to the receiver. The tube DE is called the **fall-tube** and its height is greater than the mercury barometer.



Fig, 164

Mercury is poured into the reservior A and the stop-cock S opened so that mercury flows down the tube DE. The flow is adjusted so that mercury breaks into drops in the form of pellets in DE as it falls through it drawing in air from the receiver between drop and drop. As the pellets move down the fall-tube, they carry down the air entrapped between them and the air, so carried, escapes out of the cistern in the mercury. As this process continues, more and more of air from the receiver is removed and the mercury in the cistern begins to rise up the tube DE. When the exhaustion is nearly complete, the height of the column in DE is almost the same

as that of a mercury barometer and the pellets strike the column of mercury in DE with a metallic click.

A large quantity of mercury is needed for the working of the pump. The mercury flowing into the cistern will have to be transferred to the reservoir A from time to time. The working of the pump is very slow and therefore tedious. Nowadays more efficient pumps are used which can produce the same low pressure in a comparatively much smaller time.

247. Toepler's pump

In the Toepler's pump, the fall-tube, the height of which is greater than that of the mercury barometer, is connected to an inclined wide tube A which is connected on the lower side (1) to the receiver through a valve V and (2) to a reservoir of mercury through a long vertical glass tube BC and a flexible rubber tube T. BC should also be greater than the height of the mercury barometer. The lower end of the fall-tube dips into a cistern of mercury.

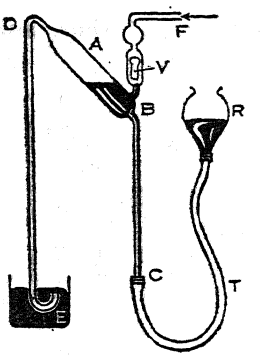


Fig. 165

The pump is worked by lowering and raising the reservoir R alternately. On lowering the reservoir the mercury in the wide tube A flows down BC and the air in the receiver enters the tube A. The reservoir is now raised when the mercury rises into A. It is prevented from entering the receiver by the valve V. When the reservoir is still further raised, the mercury in A rises still further driving the air in it down the fall-tube to escape finally through the mercury cistern.

The process is repeated and during each cycle some air in the receiver is pumped out.

The Toepler's pump produces a very high vacuum of the order of the vapour pressure of mercury, but the working is slow and laborious.

248. Cenco's hyvac pump

This is a rotary vacuum pump which produces a vacuum of the

order of 0·1 mm. very rapidly, working directly from atmospheric pressure.

It consists of a cylindrical metal casing with a shaft S passing through the centre of the casing and carrying an eccentric rotor R on it. The rotor is so arranged that it always touches the periphery of the metal casing at some point. A scraping vane V free to move up and down is kept pressed against the rotor by means of a strong spring and it separates the casing into two compartments. The casing is provided with two ports O and E on either side of the vane. The port O is connected to the receiver to be evacuated and therefore serves as the inlet port. The second port E is the exhaust and it is provided with a valve opening outwards. This arrangement is usually immersed in oil contained in a cast iron case to prevent leakage of air into the casing.

The different stages in the working of the pump can be followed as follows:

(1) When the rotor is in the position represented in Fig. 166 (a), air from the receiver expands into the space between the rotor and the casing.

(2) When the rotor turns into the position represented in Fig. (b), the air which entered the casing in (1) is compressed. Fresh air is drawn into the space above the rotor. Obviously this instalment of air cannot mix with the air drawn into the casing in (1).

(3) When the rotor takes the position in (c), the first instalment of air is further compressed. More amount of air is now drawn into the space above the rotor.

As the rotor approaches the last stage of a rotation, the air in it in the space near to the outlet gets compressed more and more so that its pressure increases sufficiently to force itself out of E. The rotor then gets back to the position in (a) when almost all the air that entered into the casing in (1) is forced out of E.

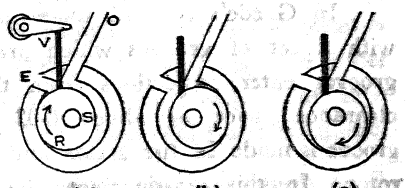


Fig. 166

The same process is then repeated in every rotation and the air in the receiver is evacuated.

In practice, two pumps are connected in series. One of them which is connected to the receiver works directly from the atmospheric pressure and produces a low vacuum and the second pump then works to produce a higher vacuum.

248. (a) Molecular pumps

Gaede and Langmuir designed molecular pumps which produce a vacuum of the order of 10^{-8} mm. to 10^{-7} of mercury working from a force-vacuum of about 2 mm. of mercury in the receiver. The molecular pumps consist of a **rotor** rotating inside a **stator**, the clearance between the two being very small, about 0·03 mm. If the gas contained in the space between the surface of the rotor and the stator is reduced to the order of about a few mm. of mercury, the mean free path of the molecules of the gas or vapour, becomes larger than the clearance and the collisions between the molecules themselves become fewer. The molecules of the gas when they strike the rotor, acquire a velocity and continue to move with that velocity as a drift velocity in the direction of ratation of the rotor. The drift velocity of the molecules has been shown by Knudsen to be equivalent to the velocity of the rotor and to remain unaffected by the subsequent collisions between the molecules themselves.

The different parts of a molecular pump are: (a) a rotor rotating about an axis inside a hermetically sealed stator, having a small clearance in between the two, (a) an inlet port and (c) an outlet port. A backing pump will have to be connected to the outlet port right from the beginning to produce the necessary fore-vacuum for the successful working of the pump.

In Gaede's type of molecular pumps, the rotor is provided with a set of grooves which are connected in series. Into these grooves enter projections from the stator leaving a very small clearance space of about 0·03 mm. In Holweck's type, the groove is made in the stator and there are no projections on the rotor. In this arrangement, the clearance space can bo made less than 0·03 mm. The molecular pumps can be used to evacuate vapours or permanent gases. But they are subject to mechanical trouble on account of the small clearance between the rotor and

stator parts. The final pressure which can be attained depends upon the fore-vacuum pressure and the velocity of rotation of the rotor. A pressure of 10^{-6} mm. of mercury can be produced with a rotor making 10,000 revolutions per minute provided the backing pump can produce a fore-vacuum of 2 mm. of mercury.

249. Diffusion pumps

Diffusion pumps work on the principle of inter-diffusion of one gas into another, the rate of which is decided by the partial pressures of the two gases. This pump works only from a low vacuum of the order of 10^{-1} mm. and it is therefore used in combination with a backing pump. The backing pump that is used generally is Cenco's hyvac pump (described in Art. 248) and it produces in the first instance the fore-vacuum required for the working of the diffusion pump.

The simple and the most efficient of the diffusion pumps is the one designed by H. Parameswaran of India in 1923 and it is nowadays as Waran's pump.

Waran's pump consists of a conical vessel A, provided with a bent glass tube BC which is fused into a wider coaxial tube BE. A side tube D connects CB to the receiver R. The tube BE branches off at the other end, one arm F being connected to the backing pump, and the other arm G being put in communication to the vessel A by means of another glass tube H. It may be noted that the tubes BE and H are inclined as shown in Fig. 167 purposely. The tube BE is surrounded by a condenser L in which cold water circulates.

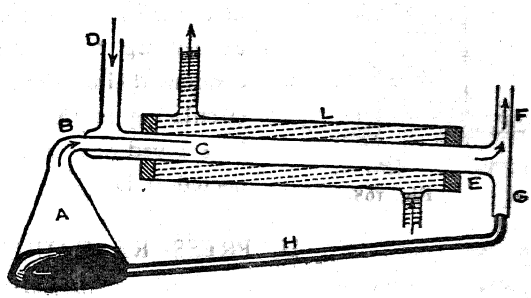


Fig. 167

The receiver is connected at D and the backing pump at F. The mercury in the vessel A is heated to boiling so that mercury vapour is profusely given out at C. If now the backing pump is set to working a few seconds, the air already present in BE will be

removed and the pressure inside becomes sufficiently less. Now, air from the receiver diffuses into mercury vapour issuing out at C and it is carried forward with the vapour. But as the mixture advances towards the other end, the cold water circulating in L condenses the vapour back to liquid mercury which flows back to the conical vessel A, through the tube H. The air that has diffused into the vapour at C is drawn out by the backing pump.

Nowadays Apiezon oil is being used instead of mercury because its vapour pressure is less than that of mercury.

These pumps are most efficient in that they produce a high vacuum of the order of 10^{-8} mm. in a comparatively short time.

However in industry, it is not possible to use pumps made of glass as it is liable to break. Steel bottle pumps which work on the same lines are therefore used. Mercury or Apiezon oil is heated to boiling in the steel bottle A. Air from the receiver, connected at B, diffuses into the vapour at C and is carried up the bottle. But the vapour is condensed into liquid by the cold water circulating in the condenser L while the air is removed out by the backing pump connected at D.

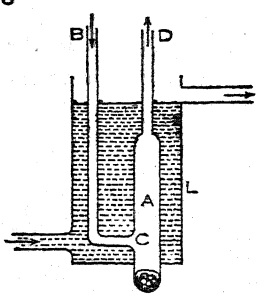


Fig. 168

H. PRESSURE GAUGES

250. Manometers

A manometer is an instrument used to measure the pressure of a gas or a liquid.

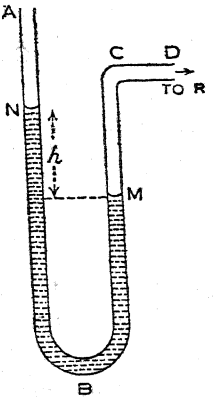
Open U-tube manometer: It consists of a U-tube open at both ends. One of the limbs BC is bent sideways so that it can be connected to a receiver, R.

A liquid (such as water or mercury) is poured down the limb AB, so that the liquid rises to some convenient height in the two limbs. When both the limbs are open to the atmosphere, the liquid stands at the same level in the two limbs.

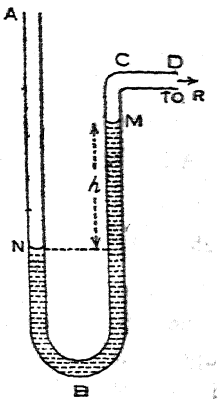
If the side-tube is now connected to the receiver, the level of the liquid M in the limb BC falls or rises and the level in AB rises or falls according as the pressure in the receiver is greater or less than the atmospheric pressure H.

If the difference between the vertical heights of M and N is h cm. and d the density of the liquid used in the manometer, the difference of pressure p is hdg dynes; if this be expressed in cm. (h_1) of mercury,

$$p = h_1 \times 13{\cdot}6 \times g.$$



Pressures higher than Atmospheric Pressure
Fig. 169 (a)



Pressures lower than Atmospheric Pressure
Fig. 169 (b)

$$\therefore h_1 \times 13{\cdot}6 \times g = hdg \text{ or } h_1 = hd / 13{\cdot}6.$$

The pressure of the gas in the receiver, P

$$= H \pm h_1 = H \pm hd / 13{\cdot}6.$$

Lighter liquids (like water or oil) are to be used for measuring small differences of pressure and mercury for large differences of pressure.

Closed U-tube manometer: A U-tube ABCD closed at one end A and filled with mercury as in Fig. 170 is used to measure high pressures. The end D is connected to the receiver, the pressure in which is required to be measured.

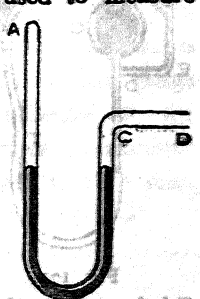


Fig. 170

To start with, the length of the air column L in AB is noted when D is open to the atmosphere. Noting the difference in levels between the two limbs and reading the mercury barometer, the pressure of the enclosed air P is determined.

If the receiver is now connected to D, the level of mercury in BC falls and that in AB rises.

If l is the length of the air column in AB and p its pressure, then according to Boyle's law

$$P \times L = p \times l$$

$$\text{or} \quad p = \frac{P \times L}{l}.$$

If h is the difference in levels of mercury, in the limbs CB and BA it is obvious that the pressure of the air in the receiver is equal to $(p + h)$.

251. McLeod Gauge

The McLeod gauge consists of a calibrated capillary glass tube AB connected to a bulb C. The volume of the bulb depends upon the order of pressures to be measured with it. The bulb is connected, in its turn, to a side tube DEF and to a reservoir of mercury R by means of a rubber tube T. A second capillary tube MN of same cross-section as AB is connected parallel to EF and is placed close to AB as in Fig. 171. This is to eliminate the capillarity effect which produces a depression in the level of mercury. A scale fixed behind the tubes helps to measure the levels of mercury in them.

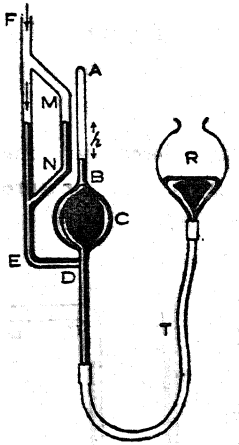


Fig. 171

McLeod gauge is used when very low pressures are to be measured. *The principle of the gauge consists in isolating a known volume V of the gas at the unknown pressure P and compressing it to a known volume v at a pressure p which can be measured.* Then according to Boyle's law, $PV = pv$ from which P is calculated.

To use the gauge, the end F is connected to the receiver before the pump is started to exhaust the air from it. The reservoir R is lowered so that the mercury in the apparatus runs down below D and the bulb C is in communication with EF and through it with the receiver. If the pump is now worked, the receiver is evacuated and the air from the gauge is also removed so that at any time the

pressure of the air in the gauge is the same as that in the receiver. To obtain the pressure P of the air in the receiver at any stage of exhaustion, the reservoir R is raised slowly so that mercury rises up and stands at D. Now the air in the tube and the bulb above D is trapped and if the reservoir is further raised the mercury rises up in the bulb compressing the trapped air. R is raised still further so that mercury enters the graduated tube AB. The volume, v, of the compressed air in AB is then read off.

As the reservoir R is raised, mercury also rises up the side-tube EF and MN also. The difference in the levels of mercury in the tubes AB and MN is noted. If h be this difference, and if p be the pressure of the air in AB, $p = P + h$, where P is the pressure of air in the receiver (and also the pressure of air in C and AB to start with).

If the initial volume of air above D (in C and AB) be equal to V, we have $P \times V = pv$

$$\text{or} \quad P \times V = (P + h)\, v$$

$$\text{or} \quad P(V - v) = hv$$

$$\text{or} \quad P = \frac{hv}{V - v}$$

from which P is calculated.

McLeod gauge is unwieldy even though it is simple and also comparatively cheap. The accuracy of the result depends upon the extent up to which Boyle's law is satisfied. If there are any traces of condensible vapours present inside C, they become a source of error. It is therefore necessary to eliminate all traces of such vapours before using the gauge. In addition, the gauge cannot correctly register pressures less than the pressure of the mercury vapour at the room temperature. If proper precautions are taken, it can be used to measure pressures correct to 10^{-4} mm. of mercury.

252. Bourdon's gauge

Bourdon's gauge consists of a circular brass tube of elliptical section closed at one end A and open at the other end C which is connected to the receiver. When the air at high pressure enters the

tube CA, thrusts produced internally tend to open out the tube and make the end A move away from C. This movement of the end A is magnified by a system of levers and finally communicated to a pointer P which moves over a scale. The scale is calibrated so that it reads pressures directly in lbs. weight per sq. inch.

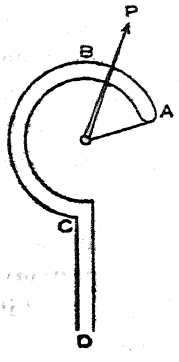


Fig. 172

Problems

1. Show that the depth of centre of pressure of a triangular lamina, when immersed vertically in a liquid of density ρ with its vertex on the surface of the liquid and its base horizontal and not subjected to any external pressure, is equal to $\frac{3}{4}$ of the vertical height of its median.

Let ABC be a triangular lamina kept with its vertex A in the surface of the liquid represented by XY and its base BC ($= a$) parallel to and at a depth b below the surface.

Consider a thin strip LM of width dh and at a depth h from the surface of the liquid.

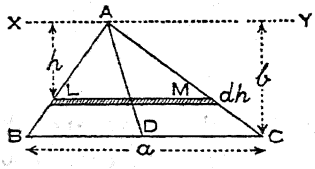


Fig. 173

Area of the strip = LM. dh.

But from similar △s ABC and ALM,

$$\frac{LM}{BC} = \frac{h}{b} \quad \text{or} \quad LM = BC\,\frac{h}{b} = \frac{ah}{b}.$$

$$\text{Area of the strip} = \frac{ah}{b} \cdot dh.$$

Pressure at the centre of the strip = $h.\ \rho.\ g$.

$$\therefore \text{Thrust on the strip} = \frac{ah}{b} \cdot dh.\ h.\ \rho.\ g = \frac{a\rho g}{b} \cdot h^2 \cdot dh.$$

Moment of the thrust about XY

$$= \frac{a \cdot \rho \cdot g}{b} \cdot h^2 \cdot dh \cdot h = \frac{a\rho g}{b} \cdot h^3 \cdot dh.$$

Considering the triangular lamina to be split up into a thin slices parallel to LM, the sum of the moments of the thrusts on

$$\text{all the strips} = \frac{a \cdot \rho \cdot g}{b} \cdot \int_0^b h^3 \cdot dh \qquad \ldots \quad (1)$$

Let H be the depth of the centre of pressure below XY.

$$\text{Resultant thrust on all the strips} = \frac{a\rho g}{b} \cdot \int_0^b h^2 \cdot dh.$$

Moment of the resultant thrust about XY

$$= \text{H.} \frac{a\rho g}{b} \cdot \int_0^b h^2 \cdot dh.$$

$$\text{Hence H.} \frac{a\rho g}{b} \int_0^b h^2 \cdot dh = \frac{a\rho g}{b} \int_0^b h^3 \cdot dh$$

$$\text{or} \quad \text{H.} \left[\frac{h^3}{3} \right]_0^b = \left[\frac{h^4}{4} \right]_0^b \quad \text{or} \quad \text{H} = \frac{3}{4} b.$$

The resultant thrust acting on each of the strips passes through the mid-point of the strip. Since all the mid-points of the strips lie on the median AD of the triangle ABC, the centre of pressure also lies on the median at a vertical depth $\frac{3}{4} b$ below the surface, i.e., at a distance $\frac{3}{4}$ AD from A.

2. *Show that the centre of pressure of a triangular strip ABC when immersed vertically in a liquid of density* ρ *with its base BC in the surface and not subjected to any external pressure is equal to* $\frac{b}{2}$, *b being the vertical depth of A below the surface of the liquid.*

Consider a thin strip LM parallel to the base at a depth h below the surface and of thickness dh.

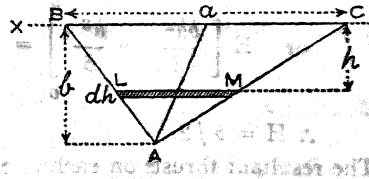


Fig. 174

Pressure acting at the centre of the strip $= h \cdot \rho \cdot g$.

Area of the strip $= \text{LM} \cdot dh$.

$$\text{But} \frac{\text{LM}}{\text{BC}} = \frac{b-h}{b} \quad \text{or} \quad \text{LM} = \frac{b-h}{b} \cdot \text{BC}.$$

$\therefore$ Area of the strip $= \frac{b-h}{b} \cdot a \cdot dh$ (if BC $= a$)

Hence thrust acting on the strip $= \frac{b-h}{b} \cdot a \cdot dh \cdot h \cdot \rho \cdot g.$

Moment of the thrust on the strip about BC

$$= h\rho g \cdot \frac{b-h}{b} \cdot a \cdot dh \cdot h = \frac{a\rho g}{b} \cdot (b-h)\, h^2 \cdot dh.$$

Considering the triangle to be divided into thin strips parallel to BC, the resultant of the moments of the thrusts on all the strips $= \frac{a\rho g}{b} \int_0^b (b-h)\, h^2 \cdot dh$ .... (1)

Resultant of thrusts acting on all the strips

$$= \int_0^b h\rho g \, \frac{b-h}{b} \cdot a \cdot dh.$$

If H is the centre of pressure, the moment of the resultant thrust about BC $= H \cdot a\, \rho\, g \int_0^b \frac{b-h}{b} \cdot h \cdot dh$ .... (2)

Equating (1) and (2)

$$H \cdot \frac{a\rho g}{b} \cdot \int_0^b (b-h)\, h \cdot dh = \frac{a\rho g}{b} \int_0^b (b-h)\, h^2 \cdot dh$$

$$\text{or} \quad H \left[\frac{bh^2}{2} - \frac{h^3}{3} \right]_0^b = \left[\frac{bh^3}{3} - \frac{h^4}{4} \right]_0^b.$$

$\therefore H = b/2.$

The resultant thrusts on each one of the strips lie at the centre of the strip and therefore lies on the median. Hence the centre of pressure also lies on the median AD at a vertical depth $\frac{b}{2}$ from the surface of the liquid.

But the thrusts are equal.

$$\therefore \frac{x}{2} \rho g a x = \left(\frac{b + x}{2} \right) \rho g a (b - x)$$

$$\text{or} \quad x^2 = (b + x)(b - x) = b^2 - x^2$$

$$\text{i.e., } b^2 = 2x^2 \quad \text{or} \quad x = \frac{b}{\sqrt{2}}.$$

2. Show that a uniform cylinder of radius r and height h floats in stable equilibrium on the surface of a liquid of density b if

$$\frac{r^2}{2} > h^2 \left(\frac{a}{b} - \frac{a^2}{b^2} \right)$$

a being the density of the material of the body.

Let ABCD be the cylinder floating on the surface of the liquid with a length x below the liquid.

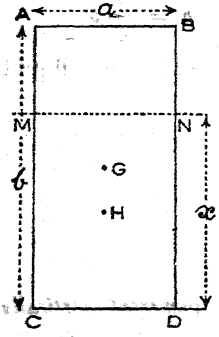


Fig. 176

Weight of the floating body, $W = (\pi r^2 h. a)$ and it acts down from its centre of gravity.

Weight of the displaced liquid $= \pi r^2 x. b$ and let it act from its centre of buoyancy H.

But the weight of the floating body = weight of the liquid displaced.

$$\therefore \pi r^2 ah = \pi r^2 xb \quad \text{or} \quad x = \frac{ah}{b}.$$

Distance between the C. G. of the cylinder and that of the liquid displaced,

$$GH = \frac{h}{2} - \frac{x}{2} = \frac{1}{2} h \left(1 - \frac{a}{b} \right) = \frac{1}{2} h \left(\frac{b - a}{b} \right).$$

If M_1 is the metacentre, then

$$HM_1 = \frac{Ak^2}{V} = \frac{A \cdot k^2}{A \cdot x} = \frac{k^2}{x} = \frac{r^2}{4} \cdot \frac{b}{ah}.$$

For stable equilibrium $HM_1 > HG$

$$\text{i.e., } \frac{r^2 b}{4ah} > \frac{1}{2} h \frac{(b - a)}{b} \quad \text{or} \quad r^2 > 2h^2 \left(\frac{a}{b} - \frac{a^2}{b^2} \right).$$

3. A load of 10 tons is moved 45 ft. across the deck of a ship of mass 8,000 tons producing a tilt of $\left(\frac{5}{8} \right)^\circ$. *Find the metacentric height.*

Total weight of hydrometer in water = 80 + 5·6 = 85·6 gm.

But weight of the liquid displaced = weight of the hydrometer together with weights added.

$\therefore v \times 1 = 85{\cdot}6.$ ... (1)

Similarly when floated in the liquid,

$v \times \rho = 80 + 12{\cdot}8 = 92{\cdot}8$ gm. ... (2)

$\therefore$ From (1) and (2), $\rho = \dfrac{92{\cdot}8}{85{\cdot}6} = 1{\cdot}084.$

6. If the density of air at N. T. P. is 0·00129 gm. / c.c., calculate the ascent corresponding to a fall of the barometer from 30 to 27 inches; the mean temperature being 25°C.

If ρ_1 and ρ_2 are the densities of air at $T_1°$ A and $T_2°$ A, $\rho_1 / \rho_2 = T_2 / T_1$

Here $\rho_1 = 0{\cdot}00129$ gm. / c.c. at 273°A

ρ_2 at 298°A is required.

$$\rho_2 = \frac{\rho_1 T_1}{T_2} = \frac{0{\cdot}00129 \times 273}{298}$$

$$= 0{\cdot}001182 \text{ gm. / c.c.}$$

From the relation,

$$\log_e \frac{\rho_1}{\rho_2} = \frac{\rho}{p} \cdot gh$$

$$h = \frac{p}{\rho g} \cdot \log_e \frac{\rho_1}{\rho_2}$$

$$= \frac{76 \times 13{\cdot}6 \times 980}{0{\cdot}001182 \times 980} \times 2{\cdot}303 \times (\log_{10} 30 - \log_{10} 27)$$

$$= 920{\cdot}2 \text{ metres.}$$

7. Assuming that a charge from 30 inches to 27 inches in the height of a mercury barometer corresponds to an altitude of 2,700 ft. find the altitude corresponding to a height of 21·87 inches of the barometer.

$p_1 = 30$ inches; $p_2 = 27$ inches.

$h = 2{,}700$ ft.

From the relation

$$\log_e \frac{p_1}{p_2} = \frac{\rho}{p} \cdot gh, \text{ we get}$$

$$2{,}700 = \frac{p}{\rho g} \cdot (\log_e 30 - \log_e 27) \quad \text{... (1)}$$

Let x be the height when the pressure is 21·87 inches.

Then $x = \frac{p}{\rho g} (\log_e 30 - \log_e 21{\cdot}87)$ ... (2)

From (1) and (2)

$$x = \frac{\log_{10} 30 - \log_{10} 21{\cdot}87}{\log_{10} 30 - \log_{10} 27} \times 2{,}700$$

$$= 8{,}106 \text{ ft.}$$

Exercises

1. (a) Define 'thrust' and 'pressure'.

(b) Show that the pressure at any point inside a homogeneous liquid at rest is the same in all directions.

2. Show that the pressure at any point on a horizontal plane inside a liquid at rest is the same.

A tank 15′ × 15′ × 6′ is filled with water. Find the thrust on a vertical side of the tank and also find where it acts.

[*Ans.* 1·688 × 10^4 lb. wt.; 4 ft. from the surface.]

3. (a) Prove that the difference in pressures between two points in a heavy homogeneous liquid at rest is proportional to the vertical distance between them.

(b) A square lamina of side a is immersed vertically with one side in the surface of a liquid of density, ρ which rests on another liquid of density σ. If the depth of the upper liquid is b ($<a$), find the thrust on the square.

Show that the thrust on the portions in the two liquids is equal when $b\rho\ (3b - 2a) = \sigma\ (a - b)^2$.

$\{$*Ans.* $\frac{1}{2} ag\ [b\rho\ (2a - b) + \sigma\ (a - b)^2]\}$

4. (a) Define 'centre of pressure'.

(b) Show that the depth of the centre of pressure for a plane area immersed vertically in a homogeneous liquid is $\frac{k^2}{h}$ where k is the radius of gyration of the area about the line where the area meets when produced into the surface and h is the depth of its centre of gravity.

5. (a) A rectangular lamina is just immersed in water vertically with one of its edges in the surface. Find the position of its centre of pressure.

(b) Find the force on a rectangular end of a tank of width 6 ft. and height 3 ft. when filled with water. Also find the centre of pressure on the end of the tank.

If it were hinged at the bottom and kept in position by a bolt at the top edge, find the force on the bolt.

[*Ans.* $1{\cdot}688 \times 10^3$ lb. wt.; 2 ft. from the top edge, 562·6 lb. wt.]

6. Find the resultant vertical thrust on the sides of a hemispherical bowl filled with a liquid and inverted over a horizontal table in terms of the thrust on the table.

[*Ans.* $\frac{1}{3}$ the thrust on the table.]

7. (a) Explain how a common hydrometer is graduated.

(b) A common hydrometer floats to the zero on its stem in water and to 100 in a liquid of density 1·25, the graduations being at equal distances. Find the division on the stem when it floats in a liquid of density 1·2. [*Ans.* 83·3]

8. (a) Define metacentre.

(b) Explain the conditions for stable and unstable equilibrium of a floating body.

9. (a) Show that if a cylinder of height h, radius r and sp. gr. s, floats in stable equilibrium in water $r^2 > 2h^2s(1 - s)$.

(b) If a common hydrometer floats to any two graduations in liquids of sp. gravities s_1 and s_2 and to a mark midway between them in a liquid of sp. gr. s, show that $\frac{2}{s} = \frac{1}{s_1} + \frac{1}{s_2}$.

10. Describe an experiment to determine the metacentric height of a ship.

A load of 50 tons when moved 30 ft. across the deck of a ship of mass 30,000 tons produces a tilt of $\frac{1}{2}°$. Find the metacentric height. [*Ans.* 5·731 ft.]

11. Write notes on :

(a) hydraulic lift (b) hydraulic pump

(c) aneroid barometer (d) centrifugal pump.

12. A square plate of side 20 cm. is lowered vertically into water with one of its edges in the surface of water. Find how a st. line parallel to the surface of water is to be drawn so that the thrusts on the two portions of the surface are equal.

[*Ans.* $10\sqrt{2}$ cm. below the surface.]

13. A plane triangular sheet of base a and altitude b is lowered into water so that its base is horizontal and its vertex is in the surface. Draw a horizontal line to divide the surface into two portions so that the thrusts on them are equal.

[*Ans.* $b/\sqrt{2}$ below the liquid surface.]

14. On one side of a rectangular vertical gate there is fresh water to a depth of 35 ft. and on the other side there is salt water. Find the depth of the salt water if the thrusts on the two sides are equal. (sp. gr. of salt water = 1·025) [*Ans.* 34·57 cm.]

15. The water on one side of a lock-gate has a depth of a ft. and on the other side, a depth of b feet. Find where the resultant thrust acts above the bottom of the water.

$$\left[\textit{Ans.} \ \frac{a^2 + ab + b^2}{3(a+b)} \right]$$

16. A Nicholson's hydrometer requires 1·5 gm. to be placed on its pan when floated in a liquid of specific gravity 0·8 and 21·5 gm. when floated to the same mark in a liquid of sp. gr. 1·2. Find the weight of the hydrometer. [*Ans.* 38·5 gm.]

17. Show that the centre of pressure on a circular lamina kept vertically with its edge touching the surface of a liquid is $\frac{5r}{4}$.

$$\left[\textit{Hint}: \ \mathrm{H} = \frac{k_0^2 + \mathrm{H}_1^2}{\mathrm{H}_1} \ ; \ \mathrm{K}_0^2 = \frac{r^2}{4} \ ; \ \mathrm{H}_1 = r. \right]$$

17 (a). Calculate the difference in heights between two stations if the atmospheric pressures at the places are 63·5 cm. and 75 cm. taking the necessary data from the tables assuming the mean temperature to be 0°C. [*Ans.* 1,330 metres.]

(b) If a change from 30 to 27 inches in the barometric height corresponds to a rise of 2,290 ft., find the altitude which corresponds to the barometric height of 24 ft. [*Ans.* 4,855 ft.]

(c) If the height of the homogeneous atmosphere is 26,000 ft., prove that the altitude above the sea level of a place at

which the barometric height is 26" when the reading at sea level is 30" is about 3,718 ft., the change in temperature being neglected.

(d) Taking the density of air as 0.00129 gm. per c.c. at N. T. P., calculate the vertical height between two places where the barometric readings are 76 and 75 cm. of mercury assuming the temperature of the atmosphere to be 15°C. (Density of mercury = 13·6 gm. per c.c.) [*Ans.* 111·0 metres.]

18. Describe the construction and action of rotary hyvac pump and explain how McLeod gauge is used to measure low pressures. (A. U. March 1961)

19. Describe the conditions of stability of floating bodies. How is the metacentric height of a ship determined? What is he importance of such a determination? (A. U. Sept., 1961)

(a) Explain how interdiffusion is utilised in production of high vacuum. (A. U. Sept., 1962)

20. Define (i) centre of pressure (ii) buoyancy and (iii) metacentre. What are the different types of equilibrium of a floating body? Calculate the thrust over a rectangular lamina immersed vertically in a homogeneous liquid with its upper edge at a depth h. (A. U. Sept., 1962)

21. Explain 'centre of buoyancy' and 'metacentre' in the case of a floating body.

State the conditions of stability of a floating body. Describe in detail, the method of graduating the stem of a common hydrometer. (A. U. Sept., 1964)

22. Discuss the forces acting on a floating body and explain why a wooden beam of rectangular section tends to float on water with its wider faces horizontal. What bearing has this problem on the design of ships? (A. U. April 1965)

23. Show how the pressure of the atmosphere varies with altitude. Calculate the altitude of a station where the atmospheric pressure is 72 cm. of mercury. (Density of mercury = 13·6 gm./c·c·; density of air at standard pressure = 1·293 gm. per litre.)
[*Ans.* 432·6 metres.] (A. U. April 1965)

24. Give an outline of the design and describe the mode of action of a high vacuum pump. How are such low pressures measured? (A. U. Sept., 1965)

25. Describe the formula for the thrust on a plane surface immersed in a liquid.

A cubical tank of side 4 ft. is placed on a horizontal floor and filled completely with water. Calculate the thrusts on the bottom and on the sides of the tank. (1 c. ft. of water weighs 62·5 lbs.)
[*Ans.* 4,000 lb. wt; 2,000 lb. wt.] (A. U. Sept., 1965)

26. Describe the construction and action of (a) a high vacuum pump and (b) McLeod gauge. (A. U. Sept., 1965)

27. Write a short note on McLeod gauge. (A. U. Sept., 1965)

28. Explain the terms 'pressure', 'thrust' and 'centre of pressure'. Find the position of the centre of pressure on a rectangular lamina immersed in a liquid with one side in the surface of the liquid. (A. U. March 1966)

29. State and explain the conditions of stability of floating bodies. Give the theory of the graduation of the common hydrometer. (A. U. April 1966)

30. State and explain the conditions of equilibrium and stability of a floating body.

When a load of 40 tons is moved 25 ft. across the deck of a ship of 20,000 tons displacement, the ship is found to tilt through an angle of 1°. Determine its metacentric height.
[*Ans.* 2·865 ft.] (A. U. Oct., 1966)

31. Describe with a neat sketch, a pump that is capable of producing pressures of the order of 10^{-6} cm. Describe a suitable gauge to measure pressures of that order. (A. U. Oct., 1966)

32. Describe with a neat diagram, the construction and principle of a high vacuum pump.

Explain how the pressure can be measured by a McLeod gauge. (A. U. April 1967)

33. Explain the terms 'pressure', 'thrust' and 'centre of pressure'. Find the position of the centre of pressure on a rectangular area immersed in a homogeneous liquid with one side

in the surface of the liquid. Calculate the total thrust on one side of a rectangular vertical dock-gate of 45 ft. width immersed in sea water to a depth of 30 ft. Given that 1 cubic foot of sea water weighs 1,025 ounces. [*Ans.* $1{\cdot}297 \times 10^7$ lb. wt.] (A. U. Oct., 1967)

34. State the conditions of stability of a floating body. Describe in detail the methods of graduating the stem of a common hydrometer.

A common hydrometer floats in water with 2 cm. of its stem above the surface. When floating in a liquid of specific gravity 1·2 it has 22 cm. of the stem above the surface. How much of the stem will be exposed above the surface, if it is floated in a liquid of specific gravity 1·1? [*Ans.* 12·9 cm. exposed.] (A. U. April 1968)

35. Obtain an expression for the difference in height between two places in terms of the barometric pressure at these two places. Hence show that if the altitude increases in arithmetic progression, the corresponding pressure decreases in geometric progression.

Calculate the altitude of the station where the atmospheric pressure is 70 cm. of mercury.

Density of mercury = 13·6 gm. / c.c.

Density of air at standard pressure = 1·293 gm. / litre.

[*Ans.* 657·2 metres.] (A. U. April 1968)

36. Write a note on 'Diffusion pump'. (A. U. Oct., 1967)

37. Define the terms 'Thrust' and 'Centre of pressure'.

Find the position of the centre of pressure of a rectangular lamina just immersed in water vertically, with one of its edges in the surface.

The water on one side of a rectangular dock-gate 11 feet wide is 10 feet deep and 20 feet on the other side. Find the resultant thrust of the water and its line of action. (A. U. Oct., 1968)

38. Describe the construction and action of a rotary hyvac pump and explain how McLeod gauge is used to measure low pressure. (A. U. Oct., 1968)

39. Write a note on 'Aneroid barometer. (A. U. Oct., 1968)

INTRODUCTORY CALCULUS

Calculus is one of the branches of Mathematics which has been found very helpful in the study of Physics. It is divided into two sections—Differential Calculus and Integral Calculus.

Constants and Variables

Any quantity which retains the same value during a set of mathematical operations is called a **constant** and a quantity which can take a set or sets of different values during such operations is called a **variable.**

Constants are usually denoted by the first letters of the alphabet a, b, c....while the variables by the later set of letters, say u, v, w, x, y, z.

Consider the relations $A = \pi r^2$ and $l = 2\pi r$. If r represents the radius of a circle, then A represents the area included by the circle and l the length of its circumference while π retains the same value in both the cases.

In these relations, the values of A and l depend upon the value given to r and if the radius of the circle is assumed to be changing, both A and l correspondingly vary. Thus A and l are said to be variables dependent upon r, and r is called the independent variable.

Consider the formula $T = 2\pi\sqrt{\frac{l}{g}}$ where T is the period of oscillation of a simple pendulum, l the length of the pendulum and g the acceleration due to gravity at the place at which the pendulum is oscillating.

It can be easily seen that there exists no relation between the length of the pendulum (l) and the acceleration due to gravity (g) and so both of these are independent variables. Thus T is the variable dependent upon both l and g. Also at a particular place g is a constant and hence T depends upon l only.

The independent variable is generally represented by x and the dependent variable by y.

Functions

Consider again the relation $A = \pi r^2$. A is called a function of r with π remaining constant. In general, if a quantity y changes with the values given to another quantity x, y is said to be **a function of** x. The quantity y need not necessarily depend upon a single variable. It may depend upon two or more variables as in the case of the volume of a cylinder which is given by $\pi r^2 h$. It is clear that in this case the volume depends on r and also on h.

The function of any variable quantity x is denoted by a single letter y. Symbols like $f(x)$, $F(x)$, $\phi(x)$ .... are also used. It should be noted that $f(x)$, $F(x)$.... do not mean that they are equal to the products of f and (x) or F and (x) respectively but stand as a whole for the function.

Some functions are explicit as in the expressions such as

$$y = 5x^2 + 6x + 7. \quad y = (2x + 3)^2 ....$$

But some others are not explicit as in the case

$$ax^2 + bxy + cy^2 = 16$$

and a function of this type is known as **an implicit function.**

Single and multi-valued functions

If for each value of x, there exists one and only one value for y, then y is called a **single valued function** and if for each value of x, there exist more than one value for y, then y is called a **multi-valued function.**

Even and Odd functions

If $f(-x) = f(x)$ then the function is called an **even function** of x, and if $f(-x) = -f(x)$ then it is said to be an **odd function** of x.

Inverse function

Consider the explicit relation $y = f(x)$. This may also be called a direct function of x. The relation suggesting x in terms of y is called the indirect or inverse function and it is denoted by $x = f^{-1}(y)$.

Thus if $y = \sin x$, then $x = \sin^{-1} y$ and so on.

Increasing and decreasing functions

If a function $f(x)$, increases (or decreases) as x increases (or decreases), it is called **increasing function** and if it decreases (or increases) as x increases (or decreases,) it is called a **decreasing function.**

Continuity of a function

If the curve representing a function is a continuous one between two points on it, then the function is said to be **continuous in that range** or **interval.** Thus $y = x^2 + 2x$ is continuous for all values of x. The function $y = \dfrac{x+3}{x-2}$ is continuous for all values of x, excepting $x = 2$ at which the curve breaks. The point at which $x = 2$ is called **the point of discontinuity of the function.**

Graphical representation of a function.

Consider a continuous function $y = f(x)$. This enables us to determine the value of y for any suggested or assumed value of x. Thus we get a series of pairs of numbers each of which may be taken as the x and y co-ordinates of a point respectively. Points are plotted on a graph paper choosing two perpendicular straight lines OX, OY as the axes of co-ordinates and the smooth free-hand curve drawn joining these points is called the **graph of the given function.** Graphical representation of functions often proves very helpful in the study of their nature.

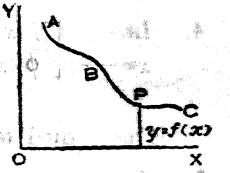


Fig. (i)

Limits

Let a variable x assume a series of values as given below:

(1) 0·9, 0·99, 0·999, 0·9999 ..... and

(2) 1·1, 1·01, 1·001, 1·0001 .....

In both the cases, the value of x gets nearer and nearer to 1 and further the difference between x and 1 can be made as small as we please. Here x is said to approach 1 or tend to the limit 1. Similarly if x tends to any constant a, then a is said to be **the limiting value** of x and is denoted by the symbol

$$x \longrightarrow a \quad \text{or} \quad \text{Lt. } x = a.$$

Consider the sum of n terms of the series

$$1 \times \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} \ldots$$

The sum of the first n terms of the series, S_n can be shown to be equal to $\left(2 - \frac{1}{2^{n-1}}\right)$. The sum S_n approaches 2 as $\frac{1}{2^{n-1}}$ tends to zero, i.e., as n tends to infinity. This may be expressed as the limit of S_n as n tends to infinity, is 2. It is often represented as

$$\underset{n \longrightarrow \infty}{\text{Lt.}} \; S_n = 2.$$

Some fundamental theorems on Limits

1. $\underset{x \longrightarrow a}{\text{Lt.}} [K.f(x)] = K. \underset{x \longrightarrow a}{\text{Lt.}} f(x)$, K being a constant.
2. $\underset{x \longrightarrow a}{\text{Lt.}} [f(x) \pm \phi(x)] = \underset{x \longrightarrow a}{\text{Lt.}} [f(x)] \pm \underset{x \longrightarrow a}{\text{Lt.}} [\phi(x)]$.
3. $\underset{x \longrightarrow a}{\text{Lt.}} [f(x) \times \phi(x)] = \underset{x \longrightarrow a}{\text{Lt.}} [f(x)] \times \underset{x \longrightarrow a}{\text{Lt.}} [\phi(x)]$.
4. $\underset{x \longrightarrow a}{\text{Lt.}} \left[\frac{f(x)}{\phi(x)}\right] = \frac{\underset{x \longrightarrow a}{\text{Lt.}} [f(x)]}{\underset{x \longrightarrow a}{\text{Lt.}} [\phi(x)]}$ provided $\underset{x \longrightarrow a}{\text{Lt.}} \phi(x)$ is not equal to zero.
5. $\underset{x \longrightarrow a}{\text{Lt.}} \frac{x^n - a^n}{x - a} = na^{n-1}$, for all rational values of 'n'.
6. $\underset{x \longrightarrow 0}{\text{Lt.}} \frac{\text{Sin } x}{x} = 1$ and $\underset{x \longrightarrow 0}{\text{Lt.}} \frac{\text{Sin } ax}{x} = a$
7. $\underset{x \longrightarrow 0}{\text{Lt.}} \frac{\tan x}{x} = 1$ and $\underset{x \longrightarrow 0}{\text{Lt.}} \frac{\tan ax}{x} = a$

} when x is measured in radians.

DIFFERENTIATION

Consider $y = f(x)$ which is a single valued, continuous function in a certain interval (range of values) of x.

Let x be given an arbitrary increment δx and the corresponding change in y be δy. It can be seen that as $\delta x \longrightarrow 0$, $\delta y \longrightarrow 0$.

Then $y + \delta y = f(x + \delta x)$

$$\therefore \delta y = f(x + \delta x) - f(x)$$

$$\therefore \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}.$$

Then the limit of $\frac{\delta y}{\delta x}$ as $\delta x \longrightarrow 0$ is called the **Differential coefficient** or the derivative of y with respect to x and is denoted by $\frac{dy}{dx}$ or $\frac{d}{dx}(y)$.

Thus $\frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lt.}} \frac{\delta y}{\delta x} = \underset{\delta x \longrightarrow 0}{\text{Lt.}} \frac{f(x + \delta x) - f(x)}{\delta x}$.

It should be carefully noted that $\frac{dy}{dx}$ is only $\underset{\delta x \to 0}{\text{Lt.}} \frac{\delta y}{\delta x}$ which should not be mistaken to be $\frac{\underset{\delta x \to 0}{\text{Lt.}} \delta y}{\underset{\delta x \to 0}{\text{Lt.}} \delta x}$ in which case it assumes the indeterminate form $\frac{0}{0}$.

Some special results

1. If $y = k$ (a constant) $\frac{dy}{dx} = \frac{d}{dx}(k) = 0$. ($k$ being any constant, there will be no change in y and therefore it does not alter whatever value x assumes).

2. If $y = ax$, then $\frac{dy}{dx} = a$.

Let y become $y + \delta y$ when x changes to $x + \delta x$.

Then $y + \delta y = a(x + \delta x)$

$$\text{or} \quad \delta y = a\,\delta x \quad \text{or} \quad \frac{\delta y}{\delta x} = a$$

$$\therefore \frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lt.}} \frac{\delta y}{\delta x} = a.$$

3. If $y = x^2$ then $\frac{dy}{dx} = 2x$

$$y + \delta y = (x + \delta x)^2 = x^2 + 2x \cdot \delta x + (\delta x)^2$$

$$\delta y = 2x \cdot \delta x + (\delta x)^2$$

$$\text{or} \quad \frac{\delta y}{\delta x} = 2x + \delta x.$$

Since y is a continuous function of x, as $\delta x \longrightarrow 0$, δy also tends to zero, i.e., $\delta y \longrightarrow 0$.

$\therefore \frac{\delta y}{\delta x}$ approaches the value $2x$ as $\delta x \longrightarrow 0$

$\therefore \frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lt.}} \frac{\delta y}{\delta x} = 2x$

$\therefore \frac{d}{dx}\left(x^2 \right) = 2x.$

4. If $y = x^3$, then $\frac{dy}{dx} = 3x^2$

Let $y + \delta y = (x + \delta x)^3$

$= x^3 + 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3$

$\therefore \delta y = 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3$

or $\frac{\delta y}{\delta x} = 3x^2 + 3x\,(\delta x) + (\delta x)^2$

Hence $\frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lt.}} \frac{\delta y}{\delta x} = 3x^2$ or $\frac{dy}{dx} = 3x^2$

5. If $y = \frac{1}{x}$ then $\frac{dy}{dx} = -\frac{1}{x^2}$

Let $y + \delta y = \frac{1}{x + \delta x}$

$$\delta y = \frac{1}{x + \delta x} - \frac{1}{x} = \frac{-\delta x}{x\,(x + \delta x)}$$

or $\frac{\delta y}{\delta x} = -\frac{1}{x(x + \delta x)}$

$\therefore \frac{\delta y}{\delta x} = -\frac{1}{x^2 + x \cdot \delta x}$

$\frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lt.}} \frac{\delta y}{\delta x} = -\frac{1}{x^2}$.

6. In general, if $y = x^n$, it can be shown that $\frac{dy}{dx} = nx^{n-1}$ for all rational values of n.

Also if $y = c.\,x^n$, $\frac{dy}{dx} = c.\,n.\,x^{n-1}$,

1. Derivative of a sum

If $y = u + v$

then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

2. Derivative of a difference

If $y = u - v$, then $\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$.

3. Derivative of a product

Let $y = uv$,

then $y + \delta y = (u + \delta u)(v + \delta v)$

$= uv + u \cdot \delta v + \delta u \cdot v + \delta u \cdot \delta v$.

$\therefore \quad \delta y = u\delta v + v\delta \cdot u + \delta u \cdot \delta v$

or $\frac{\delta y}{\delta x} = u \cdot \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} + \frac{\delta u}{\delta x} \cdot \delta v$.

Hence $\frac{dy}{dx} = u \underset{\delta x \to 0}{\text{Lt.}} \left(\frac{\delta v}{\delta x} \right) + v \underset{\delta x \to 0}{\text{Lt.}} \left(\frac{\delta u}{\delta x} \right)$

$+ \underset{\delta x \to 0}{\text{Lt.}} \left(\frac{\delta u}{\delta x} \right) \delta v = u \frac{dv}{dx} + v \cdot \frac{du}{dx}$.

4. Derivative of a quotient

Let $y = \frac{u}{v}$.

Then $y + \delta y = \frac{u + \delta u}{v + \delta v}$

and $\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v} = \frac{v\delta u - u\delta v}{v(v + \delta v)}$

$\therefore \frac{\delta y}{\delta x} = \frac{v \frac{\delta u}{\delta x} - u \frac{\delta v}{\delta x}}{v(v + \delta v)}$

and $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$.

5. Derivative of a function of a function

If $y = f(u)$ and $u = \phi(x)$ then y is said to be a function of a function

and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

The following results are to be committed to memory :

1. $\frac{d}{dx}(x^n) = nx^{n-1}$; $\frac{d}{x}\left\{(ax+b)^n\right\} = n \cdot a\,(ax+b)^{n-1}$.

2. $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$; $\frac{d}{dx}\left(\frac{1}{(ax+b)}\right) = \frac{-a}{(ax+b)^2}$.

3. $\frac{d}{dx}(\sin x) = \cos x$; $\frac{d}{dx}\left\{\sin(ax+b)\right\} = a\cos(ax+b)$.

4. $\frac{d}{dx}(\cos x) = -\sin x$; $\frac{d}{dx}\left\{\cos(ax+b)\right\}$
$= -a\sin(ax+b)$.

5. $\frac{d}{dx}(\tan x) = \sec^2 x$; $\frac{d}{dx}\left\{\tan(ax+b)\right\}$
$= a\sec^2(ax+b)$.

6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

7. $\frac{d}{dx}(\sec x) = \sec x \tan x$.

8. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

9. $\frac{d}{dx}(e^x) = e^x$; $\frac{d}{dx}(e^{ax}) = ae^{ax}$.

10. $\frac{d}{dx}(\log_{10} x) = \frac{1}{x}\log_{10} e = \frac{2{\cdot}303}{x}$.

11. $\frac{d}{dx}(\sin^n x) = n\sin^{n-1} x \cos x$.

12. $\frac{d}{dx}(\cos^n x) = -n\cos^{n-1} x \sin x$ and so on.

Successive differentiation

If y is a function of x, then its derivative $\frac{dy}{dx}$ also, in general will be a function of x.

The derivative of $\frac{dy}{dx}$ (i.e.,) $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written as $\frac{d^2y}{dx^2}$ and is called the second differential coefficient of y w. r. t. 'x'.

Similarly $\frac{d^3y}{dx^3}$ is the third differential coefficient of y w. r. t. 'x'.

For instance if $y = \sin x$

$$\frac{dy}{dx} = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2} = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right)$$

$$\frac{d^3y}{dx^3} = -\cos x = \sin\left(x + 3\frac{\pi}{2}\right)$$

in general $\frac{d^ny}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$ and so on.

APPLICATIONS OF DIFFERENTIATION

Gradient of a curve at any point (x, y) on it

Let APB be the graph of $y = f(x)$. If P be any point (x, y) on it, then the derivative of y with respect to x can be shown equal to the slope or the gradient of the curve at the point P.

If slope or the gradient of the curve at P is given by the tangent of the angle ϕ $(= 180° - \theta)$ which the tangent to the curve at P makes with the positive direction of X-axis, then it can be shown that $\tan \phi = \frac{dy}{dx}$.

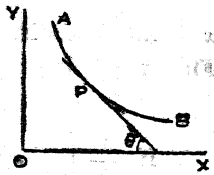


Fig. (ii)

Maxima and Minima

To obtain the maximum or minimum values of y, a function of 'x' the following procedure may be adopted:

(a) Obtain $\frac{dy}{dx}$.

(b) Solve the equation $\frac{dy}{dx} = 0$.

(c) Find out $\frac{d^2y}{dx^2}$ and find the value of $\frac{d^2y}{dx^2}$ corresponding to each value of x obtained in (b). Now three cases may arise.

1. If $\frac{d^2y}{dx^2}$ is positive, the function is minimum for that value of x.
2. If $\frac{d^2y}{dx^2}$ is negative, the function is maximum for that value of x.

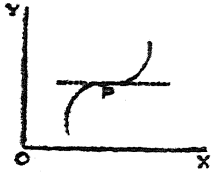


Fig. (iii)

3. If $\frac{d^2y}{dx^2} = 0$, for that value of x the tangent to the curve just crosses it and that point is called a **point of inflexion.** (vide Fig. iii).

Exercise I

Find the derivative with respect to x of the following:

1. $2x^3 - 5x^2 + 8$
2. $x^3 - 5x^2$
3. $3x^4 - x^2y + 2y^3 = 0$
4. $\text{Sin}^4 x \cos x$
5. $\text{Cos } 4x$
6. $\text{Sin } 6x$
7. $\frac{2x+3}{4x+5}$
8. $\frac{1}{\sqrt{x^2-1}}$
9. If $x = A \cos nt + B \sin nt$, prove that $\frac{d^2x}{dt^2} + n^2x = 0$.
10. An electric current flowing through a circular coil of radius a exerts a force F on a small magnet where $F \propto \frac{x}{(a^2+x^2)^{3/2}}$, x being the distance of the magnet from the centre of the coil. Find the value of x for which F is a maximum.

INTEGRATION

If $\frac{d}{dx}\left[f(x)\right] = \phi(x)$, the $\phi(x)$ is called the **derivative** of $f(x)$ with respect to x and $f(x)$ is called the **primitive** of $\phi(x)$ with respect to x. This primitive may also be called the **integral** of $\phi(x)$ with respect to x and is denoted by the symbol $\int \phi(x)\,dx$.

Thus if $\frac{d}{dx}\left[f(x)\right] = \phi(x)$, then $\int \phi(x)\,dx = f(x)$.

If c is a constant, $\frac{d}{dx}\left[f(x) + c\right] = \frac{d}{dx}f(x) + \frac{d}{dx}(c)$

$= \phi(x) + 0$

$= \phi(x)$.

Thus $\int \phi(x)\,dx = f(x) + c$ where c is a suitable constant. This constant is often called the **constant of integration.**

Indefinite integrals

Suppose for example, $\frac{dy}{dx} = 2x$, what is y?

It has been shown in differentiation that $\frac{d}{dx}(x^2) = 2x$.

$\therefore x^2$ is an indefinite integral of $2x$ with respect to x. But this is only one of the possible values of y. Since $\frac{d}{dx}(x^2 + c)$ also equals $2x$ if c is any constant.

Thus if $f(x)$ is any integral of $\phi(x)$, then $f(x) + c$ (c being a constant) is also an integral of $\phi(x)$ and it is the **general integral** and c is called the **constant of integration.**

The integral of $\phi(x)$ with respect to x is denoted by the symbol $\int \phi(x)\,dx$ which is read as "integral $\phi(x)\,dx$." The symbol dx only indicates the variable with respect to which the function is integrated.

Any equation which involves $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ etc., is called **a differential equation** and the process of determining y is called solving or integrating the equation.

The following are some of the standard integrals

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ except when $n = -1$.
2. $\int x^{-1}\,dx = \int \frac{dx}{x} = \log_e x + c.$
3. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a} + c.$
4. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c.$
5. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \text{Sin}^{-1} \frac{x}{a} + c.$
6. $\frac{dx}{\sqrt{x^2 \pm a^2}} = \log (x + \sqrt{x^2 \pm a^2}) + c.$
7. $\int \sin x\,dx = -\cos x + c.$
8. $\int \cos x\,dx = \sin x + c.$
9. $\int e^x\,dx = e^x + c.$

Definite integrals

Since $\int \phi(x)\,dx$ is a function of x, any change in the value of x results in a corresponding change in $\int \phi(x)\,dx$. The amount of change produced in the value of $\int \phi(x)\,dx$ as x changes from (say) a to b is then equal to $f(b)$, the value of $\int \phi(x)\,dx$ when $x = b$ minus $f(a)$, the value of $\int \phi(x)\,dx$ when $x = a$. This is indicated by the symbol $\int_a^b \phi(x)\,dx = f(b) - f(a)$.

It is read as the integral of $\phi(x)$ from a to b, a and b being known as the lower and upper limits of integration respectively, and $(b - a)$ is called the **range of integration.**

General theorems of integration

1. $\int k.f(x)\,dx = k.\int f(x)\,dx.$

2. If u, v, w are all functions of x, then

$$\int (u \pm v \pm w)\,dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx.$$

3. $$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

This is the formula for the integration by parts.

4. $\int f(x)\,dx = \int f(x)\frac{dx}{du}.\,du$ where x is expressed as a function of u.

For a detailed study of the subject the student is advised to refer to any treatise on calculus.

APPLICATIONS OF INTEGRATION

Cartesian co-ordinates

(i) The area 'A' included by the curve $y = f(x)$, the x axis and the ordinates at $x = a$ and $x = b$ is given by

$$A = \int_a^b y\,dx.$$

(ii) If $\bar{x}$ and $\bar{y}$ are the co-ordinates of the centre of gravity of this area they are given by

$$\bar{x} = \frac{\int_a^b x\,y\,dx}{\int_a^b y\,dx} \quad \text{and} \quad \bar{y} = \frac{\int_a^b \frac{1}{2}y^2 \cdot dx}{\int_a^b y\,dx}.$$

(iii) If this area is revolved about the x axis the volume of the solid of revolution,

$$V = \int_a^b \pi y^2\,dx \text{ and the area of the surface of revolution}$$

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

(iv) If 'S' is the length of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ then

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.\,dx.$$

(v) The co-ordinates of the centre of gravity of the area in (iii) are given by

$$\bar{x} = \frac{\int_a^b 2\pi x y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}{\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx} \text{ and } \bar{y} = 0.$$

Gravitation 114
„ Law of Universal 117
Gravitational field 118
„ Potential 119
'g' variation of 138
Gauge, McLeod 318
Gauge, Bourdon's 319
Girders 105
Graham's Law 188
Gravity Balance 141

Hooke's Law 68
Hydrometers 292
Hydrostatics 270
Hydraulic Jack 292
„ Machines 289
„ Press 289
„ Pump 291

Inertia, moment of 25
Isochronous motion 44

Jaegar 230

Kelvin's method 172
Kepler 115

Limit of Elasticity 66
Limit of Proportionality 66
Limits of σ 75
Long's method 172
Lubrication 263

Maskelyne 127
McLeod Gauge 318
Mean Free-path 156
Metacentre 296
Modulus of Elasticity 68
„ Rigidity 69
„ Young's 66
Moon and the falling body 117

Natterer 160
Neumann's Triangle 210
Neutral axis 90
„ Surface 90
Newton 114
Nicholson's Hydrometer 295

Oscillations of a loaded bar 97, 102
Osmosis 173
Osmotic Pressure 174
„ „ Laws of 178
Ostwald 258

Parallel axes Theorem 27
Pascal's Law „ 289
Pendulum, Compound 49
„ Kater's 54
„ Simple 46
Permanent Set 67
Perpendicular axes Theorem 28
Poisson's Ratio 69
Poiseuille's method 251
Poynting 136
Pressure of a gas 150
„ curvature and surface tension 212
Pumps, Centrifugal 307
„ Diffusion 315
„ Water 305
„ Hydraulic 291
„ Sprengel's 311
„ Toepler's 312

Quincke's drop method 225

Radius of gyration 25
Regnault 197
Relations between γ, n, k and σ 72
Rotation 21
Root mean square velocity 153

S. H. M. 40
„ Conditions of 42
„ Period of 41
„ Phase 46
Seprle's app. for y, n, & σ 103
Shear 65
Specific gravity 276
Spherometer 7
Spreading of liquid 209
Stokes 260
Strain 64
Stress 65
Static Torsion Apparatus 84
Surface energy 207
Surface Tension 204
„ Tension & Vapour pressure 234

Temperature & Surface Tension 232
„ and Viscosity 257
Thrust 270
„ on curved surfaces 284
Thrust & Pressure 270
„ Resultant Horizontal 284
Thrust Resultant Vertical 284
Torsion 81
Torsion Balance 89
„ Pendulum 86
Transmissibility of Fluid pressure 288
Transpiration 190

Uniform Bending Static method 96, 101
„ Dynamic method 103
Units - Systems of 2

Van der Waals' Equation 163
Vapour pressure of a solution 179
Vena Contracta 255
Viscosity 247
„ Coefficient of 247

Wilson's cloud chamber 238
Work done in Strain 76
„ in Rotation 22
„ in S. H. M. 45

Yield Point 67

ERRATA

Page	*Line*	*For*	*Read as*
22	15	angular acceleration ω	angular acceleration $\dot{\omega}$
25	19	Acceleration a	Acceleration 'a'
57	2	$T^2 (l_2 - l_2)$	$T^2 (l_1 - l_2)$
71	30	$P = \frac{\sqrt{2} \cdot T}{\sqrt{2}}$	$P = \frac{\sqrt{2} \cdot T}{\sqrt{2}}$
75	15	$3k (- 2\sigma)$	$3k (1 - 2\sigma)$
85	7	load of 100 gm.	load of 1000 gm.
89	7	$\frac{I_0 + I}{I_0} = I + \frac{I}{I_0}$	$\frac{I_0 + I}{I_0} = 1 + \frac{I}{I_0}$
98	17	$T = 2\pi\sqrt{\frac{m}{3YI/^3}}$	$T = 2\pi\sqrt{\frac{m}{3YI/l^3}}$
129	12	$\frac{g_1}{g^2}$	$\frac{g_1}{g_2}$
136	Last	$G\frac{Mm}{d^2} = \mu\theta_2$	$G\frac{Mm}{d^2} \cdot l = \mu\theta^2$
153	4	$\rho = \frac{1}{3}\rho\overline{c^2}$	$p = \frac{1}{3}\rho\overline{c^2}$
211	9	Then $\llcorner$EA	Then $\angle$EAD
271	16	the faces are uniform	the faces uniform
272	5	$P_2 \cdot a \cdot l =$	$P_2 \cdot a \cdot l =$
	8	$P_2 \cdot a \cdot l =$	$P_2 \cdot a \cdot l =$
288	20	M and N as	MN as
304	2	$(k_1 - k_1)$	$(h_1 - h_2)$
	3	$\log_e \frac{p_2}{p_2}$	$\log_e \frac{p_2}{p_3}$
310	1	P_1 is is the	P_1 is the